The order of vanishing of L-functions at the center of the critical strip

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Ad majorem Dei gloriam

Contents

Abstract

In this work we investigate the order of vanishing of $L(s,\chi)$ and $L(s,f)$ (resp. r_χ and r_f) at the centre of the critical strip, where χ is a Dirichlet character and f is a newform of weight 2 and level N ($f \in S_2(N)^{\mathrm{new}}$).

We show that on average r_f or r_χ are not large. Precisely:

(1) We generalise Murty's results on the analytic rank of $J_0(N)$ to arbitrary levels N: under the RH for all $L(s,f)$ then

$$\limsup_{N\to\infty} \frac{\displaystyle\sum_{f\in S_2(N)^{\mathrm{new}}} r_f}{\dim S_2(N)} \leq \frac{3}{2}.$$

(2) We prove that the triangle function $\max(1-|u|,0)$ is the best function when using the explicit formulas the way Brumer and Murty did.

(3) We compute an explicit bound for the second moment of the harmonic rank of $J_0(N)$. We prove that under the RH for $L(s,f)$ and the Linnik-Selberg conjecture we have the bound

$$\sum_{f\in S_2(N)^{\mathrm{new}}} \frac{r_f^2}{4\pi\langle f,f\rangle} \leq \frac{31}{12} + o(1)$$

as $N \to \infty$.

(4) We show on higher moments that

$$\sum_{f\in S_2(N)^{\mathrm{new}}} r_f^k \ll \dim S_2(N),$$

subject to the RH.

(5) We improve on the constant $\frac{3}{2}$ above to 1 when N is prime, thus proving that 50% of newforms with root number -1 have a non-vanishing L-function at $s = 1$ (conditional on the RH for $L(s, f)$ again).

(6) We show that assuming the RH for $L(s, f)$ there is, for each N sufficiently large, a newform in $S_2(N)^{\text{new}}$ so that

$$L(1 + it_0, f) = 0 \qquad \text{with} \qquad |t_0| \leq 4/\log N.$$

Résumé

Dans ce travail nous considérons l'ordre d'annulation de $L(s,\chi)$ et $L(s,f)$ (resp. r_χ et r_f) au centre de la bande critique, où χ est un caractère de Dirichlet et f une forme modulaire primitive de poids 2 et niveau N ($f \in S_2(N)^{\text{new}}$).

Nous montrons qu'en moyenne r_f et r_χ sont petits. Plus précisément:

(1) Nous généralisons les résultats de Murty sur le rang analytique de $J_0(N)$ à des niveaux N arbitraires: sous l'hypothèse de Riemann pour les $L(s,f)$ (RH), nous avons

$$\limsup_{N\to\infty} \frac{\sum_{f\in S_2(N)^{\text{new}}} r_f}{\dim S_2(N)} \leq \frac{3}{2}.$$

(2) Nous démontrons que la fonction triangle $\max(1-|u|,0)$ est la meilleure fonction test au sens de Brumer et de Murty à utiliser dans les formules explicites.

(3) Nous explicitons une borne supérieure pour le second moment du rang harmonique de $J_0(N)$. En effet sous RH et la conjecture de Linnik-Selberg, nous avons que

$$\sum_{f\in S_2(N)^{\text{new}}} \frac{r_f^2}{4\pi\langle f,f\rangle} \leq \frac{31}{12} + o(1)$$

lorsque $N \to \infty$.

(4) Nous montrons la borne (sous RH) sur les moments supérieurs:

$$\sum_{f \in S_2(N)^{\text{new}}} r_f^k \ll \dim S_2(N).$$

(5) Nous montrons qu'il est possible de choisir la constante égale à 1 au lieu de 3/2 dans le premier résultat lorsque N est premier, toujours sous RH. Comme corollaire 50% des formes primitives de signature négative ont une série L ne s'annulant pas en 1.

(6) Enfin nous faisons voir que, pour toute valeur de N suffisament grande, sous RH, il existe une $f \in S_2(N)^{\text{new}}$ telle que sa fonction L possède un zéro d'ordonnée t_0 non nulle telle que $|t_0| < 4/\log N$.

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Introduction

Let $X_0(N)$ be the modular curve of level N and $J_0(N)$ its Jacobian. A classical theorem of Mordell-Weil says that the group of rational points of $J_0(N)$ is a finitely generated abelian group of rank r. Birch and Swinnerton-Dyer have attached an L-function to $J_0(N)$. This L-function extends to an entire function and it is predicted that the order of zero of the L-function at $s = 1$ is equal to r (Birch and Swinnerton-Dyer conjecture). We will refer to the order of vanishing of this L-function as the analytic rank of $J_0(N)$.

In this work we will investigate the analytic rank of $J_0(N)$ assuming the Riemann hypothesis (RH) for L-functions of newforms of weight 2 and level N.

We will expose how one can derive the main estimates on the analytic rank of $J_0(N)$ and for instance how we were able to improve on the Brumer-Murty bound of $3/2 \dim S_2(N)$[1], where $S_2(N)$ is the space of cusps forms of weight 2 and level N for the modular group $\Gamma_0(N)$ (Murty considered the case N prime but we showed that the formula is true in general). The crucial tool used is the explicit formula of Riemann as developed by Weil, in the exposition of Mestre (cf [**Mes**]). This formula is an equality between an expression involving a function evaluated at primes on one side and an expression involving its Fourier transform computed at the zeros of an L-function on the other. Because of the inherent difficulty of estimating the location of these zeros one usually assumes the RH to handle this part. By a

[1]Murty's estimate allows one to get this bound assuming the Linnik-Selberg conjecture on Kloosterman sums in addition to the RH.

suitable choice of the test function one can then derive the Brumer-Murty estimate. We prove that unfortunately the triangle function they use is the best test function. Another feature intimately connected to these estimates is what is called the quasi-orthogonality of the Fourier coefficients of cusp forms, which can be explicitly written out in a formula involving Kloosterman sums. Finer analysis of these sums will result in further sharpening of the Brumer-Murty bound.

This problem was originally motivated by a conjecture of Greenberg [**Gre**]: let Σ be a finite set of primes and let f vary over all newforms of weight 2 for the modular group $\Gamma_0(N)$, where N is divisible only by primes of Σ. Then $\operatorname{ord}_{s=1} L(s, f) = 0$ or 1 except for at most finitely many such f's. Assuming his conjecture, Greenberg proves that for any modular elliptic curve E over $\mathbb{Q}$ with root number -1, either $E(\mathbb{Q})$ is infinite or the p-primary part of the Shafarevich-Tate group is infinite for all p where E has good ordinary reduction. This would be a precursor to a generalisation of the celebrated theorem of Kolyvagin and it would draw a step closer to the solution of the Birch and Swinnerton-Dyer conjecture since the Shafarevich-Tate group is deemed finite, thus forcing the algebraic rank of $E(\mathbb{Q})$ to be positive when its analytic rank is also positive.

The analytic rank of $J_0(N)$ can be written as a finite sum $\sum' r_f$ where $r_f = \operatorname{ord}_{s=1} L(s, f)$ is the order of vanishing of the L-function attached to a cusp form $f \in S_2(N)$. The qualitative results presented above show that under the RH for these L-functions[2] $\sum' r_f \ll \dim S_2(N)$. We prove the same for higher moments, that is $\sum' r_f^k \ll \dim S_2(N)$ for any $k \in \mathbb{N}$ (the constant involved grows like k^k).

[2]While this thesis was being written, Emmanuel Kowalski kindly provided me with a copy of his joint work with Michel [**K-M**$_1$] where they prove $\sum' r_f \ll \dim S_2(N)$ unconditionally when N is prime.

The goal of these investigations is the Brumer-Murty conjecture which asserts that $\sum' r_f \sim \dim S_2(N)/2$. Murty proved a lower bound of $\dim S_2(N)/2$, hence the interest of finding the best upper bound. This bound seems related not so much to RH but to the distribution of zeros on the critical line, given by a series of conjectures (n-correlation conjectures) stated first by Montgomery ([**Mon**]), then generalized by Rudnick and Sarnak ([**R-S**]). This can be clarified in the similar analysis in the case of the order of vanishing of Dirichlet L-functions, where a crucial role is played in estimating the error term by using the Tchebycheff ψ function.

In the same way that our methods allow to infer partial results on the analytic rank of $J_0(N)$, they also give information on the zeros close to the central zero. For example we prove that the number of zeros of all $L(s,f)$ combined ($f \in S_2(N)$) whose ordinate are $\leq 4/\log N$ is $\gg \dim S_2(N)$, hinting that maybe a positive proportion of $L(s,f)$ has zeros of low ordinate.

CHAPTER 1

L-functions

In this chapter we introduce the main objects of this thesis, namely L-functions attached to newforms and Dirichlet L-functions.

1.1. Dirichlet L-functions

1.1.1. Definitions. Let χ be a character of $\mathbb{Z}/q\mathbb{Z}$ (or character modulo q), that is a homomorphism $\chi\colon (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}^*$, which it is customary to extend to $\mathbb{Z}$ by putting it equal to zero on integers which are not coprime to q. Note that $|\chi(n)| = 1$ if $(n, q) = 1$.

The trivial character (denoted χ_0) is also called principal character modulo q.

If $q' \mid q$ it is sometimes possible to factor χ through the canonical homomorphism $\kappa\colon (\mathbb{Z}/q\mathbb{Z})^* \to (\mathbb{Z}/q'\mathbb{Z})^*$. In this case the corresponding χ' such that $\chi = \chi' \circ \kappa$ is said to induce χ. The least q' with this property is called the conductor of χ.

A primitive character is such that its conductor equals its modulus q.

Example. The principal character modulo q is induced by the character equal to 1 on $\mathbb{Z}$. The latter can be viewed as the only character modulo 1. Hence χ_0 is never primitive when $q > 1$.

Given a character χ we can attach to it an L-function

$$L(s, \chi) = \sum_{n \geq 1} \frac{\chi(n)}{n^s}.$$

Viewed as a function of the complex variable s, this function is holomorphic in the half-plane $\Re s > 1$. Furthermore it is given there as a convergent Euler product over primes

$$L(s,\chi) = \prod_{p \text{ prime}} (1 - \frac{\chi(p)}{p^s})^{-1}. \tag{1.1}$$

All these results are classical and can be found for example in [**Ser**].

These functions were the analytic tools introduced by Dirichlet to study primes in arithmetic progressions.

Example. The Riemann zeta function is the simplest example of an L-function, given by

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} = \prod_{p \text{ prime}} (1 - \frac{1}{p^s})^{-1}.$$

Its importance for the study of primes was first recognized by Euler.

1.1.2. Primitive and imprimitive L-series. Let χ be a character modulo q induced by χ' of conductor q'. We examine here the difference between the L-functions $L(s,\chi)$ and $L(s,\chi')$. By definition of induced character we have

$$\chi(n) = \begin{cases} \chi'(n) & \text{if } (n,q)=1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence we get

$$\begin{aligned} L(s,\chi) &= \prod_{p\nmid q}(1-\frac{\chi(p)}{p^s})^{-1} \\ &= \prod_{p\nmid q'}(1-\frac{\chi(p)}{p^s})^{-1} \\ &= \prod_{p\nmid q'}(1-\frac{\chi'(p)}{p^s})^{-1}\prod_{p|q}(1-\frac{\chi'(p)}{p^s}) \\ &= L(s,\chi')\prod_{p|q}(1-\frac{\chi'(p)}{p^s}). \end{aligned}$$

This holds for $\Re s > 1$ but using analytic continuation (see below) we get that

$$\operatorname*{ord}_{s=1/2} L(s,\chi) = r_\chi = r_{\chi'} = \operatorname*{ord}_{s=1/2} L(s,\chi') \tag{1.2}$$

because the finite product on the right-hand side does not vanish at $s = 1/2$.

1.1.3. Functional equation. Riemann's most original contribution was to show how $\zeta(s)$ can be analytically continued to the whole plane to a meromorphic function and how it satisfies a functional equation. This method can be extended to treat $L(s,\chi)$. To describe it we define, as in [**Dav**]

$$\mathfrak{a} = \begin{cases} 0 & \text{if } \chi(-1) = 1, \\ 1 & \text{if } \chi(-1) = -1. \end{cases}$$

We now assume χ to be primitive, for otherwise the functional equation becomes complicated. Define

$$\xi(s,\chi) = (\frac{\pi}{q})^{-\frac{1}{2}(s+\mathfrak{a})}\Gamma\left(\frac{1}{2}(s+\mathfrak{a})\right)L(s,\chi)$$

and introduce the Gauss sum

$$\tau(\chi) = \sum_{m=1}^{q}\chi(m)e^{\frac{2\pi i m}{q}}.$$

Then ξ satisfies the equation

$$\xi(1-s,\bar{\chi}) = \frac{i^{\mathfrak{a}} q^{\frac{1}{2}}}{\tau(\chi)} \xi(s,\chi) \tag{1.3}$$

where $\bar{\chi}$ is the complex conjugate of χ.

Remark. It is particularly interesting to examine the case of real characters (e.g. $\zeta(s)$). In this case an analysis of the sign of the Gauss sum (cf [**Dav**, p. 13]), leads to the simple functional equation

$$\xi(1-s,\chi) = \xi(s,\chi). \tag{1.3′}$$

This shows a symmetry role played by the point $s = 1/2$. Also, because of the invariance of the last equation under complex conjugation, the line $\Re s = 1/2$ seems to play an important role. Note that in this case $\operatorname{ord}_{s=1/2} L(s,\chi)$ is even.

Remark. The strip $0 \leq \Re s \leq 1$ is called the critical strip. From the Euler product expansion (1.1) it follows easily that $L(s,\chi) \neq 0$ in $\Re s > 1$, hence $\xi(s,\chi) \neq 0$ there. From the functional equation we infer that $\xi(s,\chi) \neq 0$ in $\Re s < 0$, hence its zeros can only possibly lie in the critical strip. It is a non-trivial fact to show that $L(s,\chi) \neq 0$ for $\Re s = 1$.

One final point of consideration. We did not actually show that ξ can be continued inside the critical strip. This follows by applying partial summation to get

$$\begin{aligned} L(s,\chi) &= \sum_{n\geq 1} \frac{\chi(n)}{n^s} \\ &= \lim_{x\to\infty} \left(\frac{\sum_{n\leq x}\chi(n)}{x^s} + s\int_1^x \frac{\sum_{n\leq t}\chi(n)}{t^{s+1}}\,dt \right) \\ &= s\int_1^\infty \frac{\sum_{n\leq t}\chi(n)}{t^{s+1}}\,dt \quad \text{if } \Re s > 1. \end{aligned}$$

Notice that from Theorem 2.4 we have $\sum_{n\le x}\chi(n) = \delta_{\chi_0}x + O(1)$, where $\delta_{\chi_0} = \prod_{p|q}(1-1/p)$ if $\chi=\chi_0$ and zero otherwise. Hence we get

$$\begin{aligned} L(s,\chi) &= s\int_1^\infty \frac{\delta_{\chi_0}t + O(1)}{t^{s+1}}\,dt \\ &= \delta_{\chi_0}\frac{s}{s-1} - s\int_1^\infty \frac{O(1)}{t^{s+1}}\,dt. \end{aligned}$$

The integral defines an analytic function in $\Re s > 0$, hence the previous formula gives analytic continuation inside the critical strip, with possibly a simple pole at 1 with residue δ_{χ_0}.

Remark. Actually when $L(s,\chi)=\zeta(s)$, then

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{\{t\}}{t^{s+1}}\,dt, \tag{1.4}$$

where $\{t\}$ is the fractional part of t. From (1.4) we infer that $\zeta(1/2)<0$.

1.2. Modular L-functions

Bearing the same name, these functions also share some features with the Dirichlet L-functions. We will focus on the differences. All the material discussed here can be found for example in [**Kna, Shi**$_1$].

1.2.1. Modular curves. Let $\mathcal{H} = \{z\in\mathbb{C}\colon \Im z > 0\}$ be the upper half-plane. The group $SL_2(\mathbb{Z})$ acts on $\mathcal{H}$ by Möbius transformation:

$$\gamma\colon z\mapsto \frac{az+b}{cz+d}, \quad \gamma = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL_2(\mathbb{Z}).$$

We can extend this action to $\mathcal{H}^* = \mathcal{H}\bigcup\mathbb{Q}\bigcup\{\infty\}$ by continuity (we call $\mathbb{Q}\bigcup\{\infty\}$ the set of cusps). Hence $\gamma(\infty)=a/c$ in the above notation.

Let $N>0$ be an integer. We introduce the Hecke group $\Gamma_0(N)$ as the subgroup of $SL_2(\mathbb{Z})$ consisting of matrices γ as before such that $N\mid c$. The complex curve obtained through the action of $\Gamma_0(N)$ on $\mathcal{H}^*$ is the modular curve $X_0(N)$. It is a moduli space classifying elliptic curves over $\mathbb{C}$ having a distinguished cyclic subgroup of order N. The modular curve $X_0(N)$ can be given the structure of a compact connected Riemann surface. It contains a lot of arithmetical information. We will see some of it as we discuss its holomorphic differentials.

1.2.2. Modular Forms and Cusp Forms. We define on the set of holomorphic functions on $\mathcal{H}$ an action of $\Gamma_0(N)$ given by

$$f|_\gamma(z) = \frac{1}{(cz+d)^2} f(\frac{az+b}{cz+d}).$$

We want to look at functions f satisfying

$$f = f|_\gamma \quad \text{for } \gamma \in \Gamma_0(N). \tag{1.5a}$$

Let κ be a cusp and $\sigma \in SL_2(\mathbb{Z})$ be such that $\sigma(\infty) = \kappa$. If f satisfies (1.5a) then $f|_\sigma$ is invariant under $\Gamma = \sigma^{-1}\Gamma_0(N)\sigma$. It is easy to see that Γ has finite index in $SL_2(\mathbb{Z})$, hence there exists an integer $M \geq 1$ such that

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^M = \begin{pmatrix} 1 & M \\ 0 & 1 \end{pmatrix} \in \Gamma.$$

This means that $f|_\sigma(z+M) = f|_\sigma(z)$ for all $z \in \mathcal{H}$ and hence that $f|_\sigma$ has a "Fourier expansion" at κ given for example by

$$f|_\sigma(z) = \sum_{-\infty<n<\infty} a_n e^{\frac{2\pi i n z}{M}}.$$

We say that f is holomorphic at the cusp κ if

$$a_n = 0 \quad \text{for } n < 0 \tag{1.5b}$$

and that it vanishes at κ if

$$a_n = 0 \quad \text{for } n \leq 0. \tag{1.5c}$$

Example. If f satisfies condition (1.5a) then $f(z+1) = f(z)$ and $M = 1$ for the cusp ∞. The Fourier expansion at infinity is also called q-expansion, where $q = e^{2\pi i z}$.

Definition 1.1. A holomorphic function on $\mathcal{H}$ is called a modular form (resp. cusp form) of weight 2 and level N (for $\Gamma_0(N)$) if it satisfies (1.5a) and (1.5b) (resp. (1.5a) and (1.5c)) at all the cusps.

Remark. The vector spaces of modular and cusp forms of weight 2 and level N are denoted respectively by $M_2(N)$ and $S_2(N)$.

The holomorphic differentials on $X_0(N)$ form a $\mathbb{C}$-vector space of dimension g, the genus of $X_0(N)$. They can be explicitly described as the forms $f(z)\,dz$, where f is a cusp form of weight 2 for $\Gamma_0(N)$. There is a formula which gives this dimension, again given in [**Kna**, p. 272]. For our purposes it suffices to mention that $\dim S_2(N) \sim (N/12)\prod_{p|N}(1+1/p)$. As a consequence we have that, if $J_0(N) = \mathrm{Jac}\big(X_0(N)\big)$, then $\dim\big(J_0(N)\big) \sim N/12$ when N is prime.

1.2.3. Hecke operators and Petersson Scalar Product. We define the Hecke operators as a set of linear endomorphisms of $M_2(N)$ indexed by $n \in \mathbb{N}$. Let $q = e^{2\pi i z}$ and let $f \in M_2(N)$ have q-expansion $f(z) = \sum_{n=0}^{\infty} c_n q^n$. Then $\mathbf{T}_m f(z) = \sum_{n=0}^{\infty} b_n(m) q^n$, where

$$b_n(m) = \begin{cases} c_0 \sum_{\substack{d|m,\ d>0 \\ (d,N)=1}} d & \text{if } n = 0 \\ c_m & \text{if } n = 1 \\ \sum_{\substack{d|(n,m) \\ (d,N)=1}} d\, c_{nm/d^2} & \text{if } n > 1. \end{cases} \tag{1.6}$$

REMARK. Notice that $\mathbf{T}_m\big|_{S_2(N)} \in \mathrm{End}\big(S_2(N)\big)$.

We define an hermitian product on $S_2(N)$. It can be shown that if $z = x + iy$ with $x, y \in \mathbb{R}$, then $f \in S_2(N)$ implies $|f(z)|\, y$ is bounded and invariant under $\Gamma_0(N)$. Furthermore the differential $\frac{dx \wedge dy}{y^2}$ is $\Gamma_0(N)$-invariant, hence the following definition.

DEFINITION 1.2. For $f, g \in S_2(N)$ the Petersson inner product is defined as

$$\langle f, g\rangle = \int_{X_0(N)} f(z)\overline{g(z)}\, dx\, dy$$

We have the following nice fundamental theorem (see [**Shi**$_1$]).

THEOREM 1.1 (Hecke). *The algebra* $\mathrm{End}\big(S_2(N)\big)$ *is commutative. Furthermore, the operators* $\mathbf{T}_m$ *are self-adjoint with respect to the Petersson inner product when* $(m, N) = 1$.

From this follows that we can simultaneously diagonalise all the $\mathbf{T}_m$'s with $(m, N) = 1$. Unfortunately we cannot usually diagonalise also the $\mathbf{T}_m$'s with $(m, N) > 1$ on $S_2(N)$, but we can on a smaller space, called the space of newforms or $S_2^{\mathrm{new}}(N)$. We call eigenforms for $\Gamma_0(N)$ (resp. weak eigenforms for $\Gamma_0(N)$) such forms which are eigenvectors for all the $\mathbf{T}_m$ (resp. $\mathbf{T}_m$ with $(m, N) = 1$).

1.2.4. The space $S_2^{\mathbf{new}}(N)$. It plays the same role to modular forms as primitive characters to Dirichlet characters. Indeed we have a nice description of the orthogonal complement of $S_2^{\text{new}}(N)$ in $S_2(N)$, called $S_2^{\text{old}}(N)$.

THEOREM 1.2. *If $d_1 d_2 \mid N$ and $f(z)$ is a weak eigenform for $\Gamma_0(\frac{N}{d_1})$ then $f(d_2 z)$ is a weak eigenform for $\Gamma_0(N)$ with same eigenvalues (for $\mathbf{T}_m$, $(m,N)=1$).*

The $\mathbb{C}$ span of all such forms arising from eigenforms of lower level is $S_2^{old}(N)$.

Next we state the famous result of Atkin-Lehner [**A-L**].

THEOREM 1.3 (Atkin-Lehner). *Each eigenspace for all $\mathbf{T}_m$'s of $S_2^{new}(N)$ is one dimensional.*

PROOF. This amounts to proving that the set of eigenvalues a_m uniquely determines the eigenform up to scalars.

Let the eigenform $f \in S_2^{\text{new}}(N)$ have q expansion $f(z) = \sum_{n=1}^{\infty} c_n q^n$. Then from (1.6) we get that $c_n = a_n c_1$. In other terms, if $f \neq 0$ then $c_1 \neq 0$, hence putting $c_1 = 1$ we get that $c_n = a_n$. □

THEOREM 1.4. *Let $f(z) = \sum_{n=1}^{\infty} a_n q^n$ be an eigenform for $\Gamma_0(N)$ with $a_1 = 1$. We have the following identities on the a_n's:*

$$
\begin{aligned}
& a_{p^r} a_p = a_{p^{r+1}} + p a_{p^{r-1}} && \text{for } p \text{ prime, } p \nmid N \\
& a_{p^r} = a_p^r && \text{for } p \text{ prime, } p \mid N \\
& a_m a_n = a_{mn} && \text{if } (m,n) = 1.
\end{aligned}
\tag{1.7}
$$

1.2.5. L-function. Let $f \in S_2(N)$. Then from the boundedness of $|f(z)|\, y$ on $\mathcal{H}$ (cf. above), we get that if $f(z) = \sum_{n=1}^{\infty} a_n q^n$ then $a_n = O(n)$. Hence the

L-function

$$L(s, f) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

converges for $\Re s > 2$ to a holomorphic function. Actually it also follows from the boundedness of $|f(z)|\,y$ that

$$\sum_{n \le x} |a_n| \ll x^{3/2}$$

hence that the Dirichlet series for $L(s, f)$ converges absolutely in the half-plane $\Re s > 3/2$ (cf. [**Iwa**$_2$, p. 71, cor. 5.2]).

If $f \in S_2^{\text{new}}(N)$ is an eigenform with $a_1 = 1$ then from (1.7) $L(s, f)$ has the Euler product

$$L(s, f) = \prod_{p|N} (1 - a_p p^{-s})^{-1} \prod_{p \nmid N} (1 - a_p p^{-s} + p^{1-2s})^{-1}.$$

Remarks.

(1) The characteristic polynomial $x^2 - a_p x + p$ can be factored as $(x - \alpha_p)(x - \bar{\alpha}_p)$ where $\alpha_p \in \mathbb{C}$. This is the analogue of the Riemann hypothesis for zeta functions of curves over finite fields, proven by Deligne in 1973. It follows that $|a_p| \le 2\sqrt{p}$ when $p \nmid N$, hence that the Euler product converges for $\Re s > 3/2$.

(2) If $p^2 | N$, then Atkin and Lehner prove in [**A-L**] that $a_p = 0$; if $p \| N$, they prove that $a_p = \pm 1$.

(3) Another interesting feature of the eigenvalues a_n is that the compositum of the fields $\mathbb{Q}(a_p)$ is a real field of finite degree over $\mathbb{Q}$. Also, a_n is an algebraic integer (cf. [**Shi**$_1$]).

1.2.6. Functional equation. We define the Atkin-Lehner involution as the transformation $\mathbf{W}_N$ given by the matrix $\left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$. Then $\mathbf{W}_N \colon f \mapsto f\big|_{\mathbf{W}_N}$ carries $S_2(N)$ into itself (also true for $M_2(N)$). Also $\mathbf{W}_N$ is self-adjoint with respect to the

Petersson product and it commutes with all the $\mathbf{T}_n$, $(n, N) = 1$. Therefore we have the decomposition $S_2(N) = S_2^+(N) \bigoplus S_2^-(N)$ into eigenspaces to the eigenvalues $+1$ and -1 respectively (note that $\mathbf{T}_n$ fixes these subspaces for all n). Since $\mathbf{W}_N$ is diagonalisable on $S_2(N)$, the eigenspaces consist of eigenvectors for the relative eigenvalue. For such forms we have a nice functional equation.

THEOREM 1.5 (Hecke). *Let $f \in S_2^{\pm}(N)$. Then $L(s, f)$ extends to an entire function. Moreover, the function*

$$\Lambda(s, f) = \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s) L(s, f)$$

satisfies the functional equation

$$\Lambda(s, f) = \mp \Lambda(2 - s, f)$$

REMARK. It follows from the theorem of Atkin-Lehner that if $f \in S_2^{\text{new}}(N)$, then f is an eigenvector for $\mathbf{W}_N$. In view of Deligne's theorem (formerly Ramanujan-Petersson conjecture) and the functional equation, the Euler product ensures that the zeros of $\Lambda(s, f)$ all lie in the critical strip $1/2 \leq \Re s \leq 3/2$. The centre of the strip is 1, as opposed to $1/2$ in the case of Dirichlet L-functions, but the width is the same.

Notice that analytic continuation inside the critical strip is obtained from the proof of Theorem 1.5.

1.3. The analytic rank of $J_0(N)$

1.3.1. The Jacobian $J_0(N)$. We review here some brief facts about the Jacobian of $X_0(N)$. We already know that $X_0(N)(\mathbb{C})$ is a compact connected Riemann surface of genus g equal to $\dim S_2(N)$. Let $\{\omega_1, \dots, \omega_g\}$ be a basis of the holomorphic differentials on $X_0(N)$ and $\{\gamma_1, \dots, \gamma_{2g}\}$ be a basis of $H_1\big(X_0(N)(\mathbb{C}), \mathbb{Z}\big) \cong \mathbb{Z}^{2g}$. Then it is a well-known theorem of Abel that the $g \times 2g$ matrix

$$(\mathbf{v}_1, \dots, \mathbf{v}_{2g}) = \begin{pmatrix} \int_{\gamma_1} \omega_1 & \dots & \int_{\gamma_{2g}} \omega_1 \\ \vdots & \vdots & \vdots \\ \int_{\gamma_1} \omega_g & \dots & \int_{\gamma_{2g}} \omega_g \end{pmatrix}$$

has rank g, moreover that

$$\Lambda^{(N)} = \mathbb{Z}\mathbf{v}_1 + \dots + \mathbb{Z}\mathbf{v}_{2g}$$

is a lattice inside $\mathbb{C}^g$, called the lattice of periods.

Definition 1.3. The Jacobian of $X_0(N)$, denoted by $J_0(N)$, is the abelian variety

$$J_0(N) = \mathbb{C}^g / \Lambda^{(N)}.$$

This definition shows that $J_0(N)(\mathbb{C})$ has a natural structure of an abelian group. A classical theorem of Mordell-Weil states that if K is a finite extension of $\mathbb{Q}$, the set $J_0(N)(K)$ of K-rational points on $J_0(N)$ is a finitely generated abelian group, hence has finite rank. We are interested here in finding a good upper bound on the rank when $K = \mathbb{Q}$. Unfortunately, this is very difficult in this setting, so we will work on the more manageable notion of analytic rank.

1.3.2. Definition of analytic rank. It is also possible to define the L-function of $J_0(N)$. It is a theorem of Shimura (cf. [**Shi**$_1$]) that when N is prime

$$L(s, J_0(N)) = \prod_{f \text{ eigenform of } S_2^{\text{new}}(N)} L(s, f)$$

In general the above product defines the L-function of $J_0^{\mathrm{new}}(N)$. Let $\sum'$ denote the sum over eigenforms of weight 2 and level N, for N prime. Then the analytic rank of $J_0(N)$ is defined as

$$\text{analytic rank}\big(J_0(N)\big) \stackrel{\text{def}}{=} \operatorname*{ord}_{s=1} L(s, J_0(N)) = \sum{}' \operatorname*{ord}_{s=1} L(s,f) = \sum{}' r_f$$

1.3.3. Ram Murty's lower bound. We sketch here Murty's proof in [**Mur**$_1$] to show the following result. This proof goes back to an argument of Mazur [**Maz**].

THEOREM 1.6 (Murty). *The analytic rank is bounded below by*

$$\text{analytic rank}\big(J_0(N)\big) \geq \frac{1}{2} \dim S_2(N) + O(\sqrt{N})$$

PROOF. The Atkin-Lehner involution $\mathbf{W}_N$ can be viewed as a morphism

$$\mathbf{W}_N \colon X_0(N) \longrightarrow X_0(N),$$

giving rise to a natural quotient map $X_0(N) \longrightarrow X_0(N)^+ = X_0(N)/\mathbf{W}_N$ of degree 2. Note that the holomorphic differentials of $X_0(N)^+$ are described by $S_2(N)^+$, hence the genus of $X_0(N)^+$ equals $\dim S_2(N)^+$. By the Riemann-Hurwitz formula we find

$$\dim S_2(N) = 2 \dim S_2(N)^+ + s$$

where s is the number of fixed points of $\mathbf{W}_N$. Such a point is represented by a cyclic isogeny of degree N

$$E \xrightarrow{\pi} E'$$

which is isomorphic to its dual

$$E' \xrightarrow{\hat{\pi}} E.$$

It follows that $E \simeq E'$ and that π is a non-trivial endomorphism of E of degree N (because the endomorphism $[n]$ is not cyclic when $n > 1$). It is a known fact that

End(E) is in this case an order in an imaginary quadratic field and that the images of π and $\hat{\pi}$ are complex conjugate of each other. Since

$$\hat{\pi} \circ \pi = \epsilon \, [\deg \pi] = \epsilon N,$$

where ϵ is a unit in the imaginary quadratic field $\mathbb{Q}(\sqrt{-N})$, we deduce that E has complex multiplication by $\mathbb{Z}[\sqrt{-N}]$ (easily seen if $\epsilon = \pm 1$ but a little computation shows it's true in general). But the number of such E's is at most $O(\sqrt{N})$, by well-known estimates for the class number of $\mathbb{Q}(\sqrt{-N})$ (cf. [**Dav**, p. 50]). Hence we get

$$\dim S_2^+(N) = \frac{1}{2} \dim S_2(N) + O(\sqrt{N})$$

From Theorem 1.5 if $f \in S_2^+(N)$, then r_f is odd, whence the bound. $\square$

CHAPTER 2

Explicit Formulas, orthogonality and main conjectures

2.1. Explicit Formulas

In his only paper in number theory, Riemann [**Rie**] was able to explain the fundamental relation that runs between primes and zeros of the zeta function. The identity, valid for $c > 0$

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^s \frac{ds}{s} = \begin{cases} 0 & \text{if } 0 < y < 1 \\ \frac{1}{2} & \text{if } y = 1 \\ 1 & \text{if } y > 1 \end{cases}$$

together with the fact that

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

whenever $\Re s > 1$ shows that

$$\psi_0(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} -\frac{\zeta'}{\zeta}(s)\,\frac{x^s}{s}\,ds \tag{2.1}$$

where $\psi_0(x) = \psi(x) = \sum_{n\le x}\Lambda(n)$ when x is not a prime power and $\psi_0(x) = \psi(x) - \Lambda(x)/2$ when it is. Then, without justifying it, Riemann shifted integration contour by letting $c \to -\infty$. Computing residues he deduced his explicit formula

$$\psi_0(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \frac{\zeta'}{\zeta}(0) - \frac{1}{2}\log(1 - x^{-2}). \tag{2.2}$$

Weil reproduced the same type of formula relating zeros of L-functions to primes in a smoother version. Indeed if we take a function Φ which decreases rapidly enough as $\Im s \to \infty$,then the function

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} -\frac{\zeta'}{\zeta}(s)\Phi(s\log x)\,ds$$

will be sufficiently regular in x. The explicit formula is the key step to the "complex" proofs of the prime number theorem.

2.1.1. Statement of the Riemann-Weil explicit formulas theorem. We refer to [**Mes**] for details. The following exposition is taken out of [**M-S**].

Let M and M' be two non-negative integers, A and B two positive real numbers, $(a_i)_{1\leq i\leq M}$ and $(a_i{}')_{1\leq i\leq M}$ two sequences of non-negative real numbers such that $\sum_{i=1}^{M} a_i = \sum_{i=1}^{M} a'{}_i$. Finally, let $(b_i)_{1\leq i\leq M}$ and $(b_i{}')_{1\leq i\leq M}$ be two sequences of complex numbers with non-negative real part.

Suppose there exist two meromorphic functions Λ_1 and Λ_2 verifying the following conditions:

(1) There exists $w \in \mathbb{C}^*$ so that $\Lambda_1(1-s) = w\Lambda_2(s)$,
(2) Λ_1 and Λ_2 have only a finite number of poles,
(3) For $i = 1$ or 2, Λ_i minus its singular terms is bounded inside any vertical strip of the form

$$-\infty < \sigma_0 \leq \Re s \leq \sigma_1 < +\infty$$

(4) There exists $c \geq 0$ such that, for $\Re s > 1 + c$ we have:

$$\Lambda_1(s) = A^s \prod_{j=1}^{M} \Gamma(a_j s + b_j) \prod_p \prod_{i=1}^{M'} \big(1 - \alpha_i(p)p^{-s}\big)^{-1}$$

$$\Lambda_2(s) = B^s \prod_{j=1}^{M} \Gamma(a_j{}' s + b_j{}') \prod_p \prod_{i=1}^{M'} \big(1 - \beta_i(p)p^{-s}\big)^{-1}$$

where p runs over all the prime numbers and $\alpha_i(p)$, $\beta_i(p)$ are complex numbers of modulus $\leq p^c$.

Let $F\colon \mathbb{R} \to \mathbb{R}$ be a function satisfying the following conditions:

(1) There exists $\epsilon > 0$ such that $F(x)\exp\big((1/2 + c + \epsilon)x\big)$ is integrable and has bounded variation, the value at each point being the average of the left-hand limit and the right-hand limit.

(2) $\big(F(x) - F(0)\big)/x$ has bounded variation.

We define

$$\Phi(s) = \int_{-\infty}^{+\infty} F(x)e^{(s-1/2)x}\,dx$$

and

$$I(a,b) = -\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Phi(s)\frac{\Gamma'}{\Gamma}(as+b)\,ds$$

$$J(a,b) = -\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Phi(1-s)\frac{\Gamma'}{\Gamma}(as+b)\,ds.$$

We then have the following theorem:

Theorem 2.1 (Riemann-Weil explicit formulas). *Notations as above, the following formula is true:*

$$\sum_{\rho}\Phi(\rho) - \sum_{\mu}\Phi(\mu) + \sum_{i=1}^{M} I(a_i,b_i) + \sum_{i=1}^{M} J({a_i}',{b_i}') =$$

$$F(0)\log(AB) - \sum_{p,i,k\geq 1}\big(\alpha_i^k(p)F(k\log p) + \beta_i^k(p)F(-k\log p)\big)\frac{\log p}{p^{k/2}}$$

where ρ (resp. μ) runs over the zeros (resp. the poles) of Λ_1 in the critical strip $-c \leq \Re s \leq 1 + c$, each of them counted with multiplicity.

Remark. As in the original Riemann sum, the expression $\sum_\rho \Phi(\rho)$ is to be understood as $\lim_{T\to+\infty}\sum_{|\Im\rho|<T}\Phi(\rho)$.

PROOF. Cf [**Mes**] or [**R-S**, pp. 277–278]. The idea is the same as Riemann's. For simplicity we will assume that the test function F is compactly supported, as will be the case in the applications and that it is also $C^\infty(\mathbb{R})$. Start from

$$\frac{1}{2\pi i}\int_{(1+2c)} \Phi(s)\frac{\Lambda_1{}'}{\Lambda_1}(s)\,ds$$

and shift the line of integration to $\Re s = -2c$. Because Φ is basically the Fourier transform of a regular function, it is rapidly decreasing in $\Im s$, uniformly in any vertical strip, hence the contour shifts are legitimate. We then pick up poles and zeros of Λ_1'/Λ_1 inside the critical strip and get, using the functional equation

$$\frac{1}{2\pi i}\int_{(1+2c)} \Phi(s)\frac{\Lambda_1{}'}{\Lambda_1}(s)\,ds + \frac{1}{2\pi i}\int_{(1+2c)} \Phi(1-s)\frac{\Lambda_2{}'}{\Lambda_2}(s)\,ds = \sum_\rho \Phi(\rho) - \sum_\mu \Phi(\mu).$$

We are left to prove that

$$\frac{1}{2\pi i}\int_{(1+2c)} \Phi(s)\frac{\Lambda_1{}'}{\Lambda_1}(s)\,ds = -\sum_{p,i,k\geq 1} \alpha_i^k(p)F(k\log p)\frac{\log p}{p^{k/2}} - \sum_{i=1}^{M} I(a_i,b_i) + F(0)\log A$$

because the other part follows by a symmetric argument.

Ultrametric part. If we define

$$L(s) = \prod_p \prod_{i=1}^{M'} \big(1-\alpha_i(p)p^{-s}\big)^{-1}$$

it then follows after taking logarithmic derivatives that

$$\frac{L'}{L}(s) = -\sum_{i=1}^{M'} \sum_{\substack{p \text{ prime} \\ k\geq 1}} \frac{(\log p)\alpha_i^k(p)}{p^{ks}} = -\sum_{n\geq 1}\frac{c_n}{n^s}.$$

Hence

$$\begin{aligned}\frac{1}{2\pi i}\int_{(1+2c)} \Phi(s)\frac{L'}{L}(s)\,ds &= -\sum_{n=1}^{\infty}\frac{c_n}{\sqrt{n}}\frac{1}{2\pi}\int_{-\infty}^{\infty}\phi(t)e^{-it\log n}\,dt \\ &= -\sum_{n\geq 1}\frac{c_n}{\sqrt{n}}F(\log n).\end{aligned}$$

where $\phi(t) = \Phi(1/2+it)$.

Archimedean part. The proof is clear, since $\sum_{i=1}^{M} I(a_i,b_i) + F(0)\log A$ is the missing term. This is clear for $F(0)\log A$, as for the integral terms it suffices to note that

$$\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\Phi(s)\frac{\Gamma'}{\Gamma}(a_i s+b_i)\,ds = \frac{1}{2\pi i}\int_{(1+2c)}\Phi(s)\frac{\Gamma'}{\Gamma}(a_i s+b_i)\,ds$$

because since $\Re b_i \geq 0$, no poles are picked up in shifting the line of integration to $\Re s = 1/2$. □

2.2. Application to $L(s,f)$ and $L(s,\chi)$

2.2.1. Explicit formula for $L(s,\chi)$. Let χ be a primitive character to the modulus q. After defining $\Lambda_1(s) = \xi(s,\chi)$ and $\Lambda_2(s) = \xi(s,\bar{\chi})$ we see from (1.3) that the functions satisfy the hypothesis of Theorem 2.1 with $M = M' = 1$, $a_i = 1/2$, $b_i = \mathfrak{a}/2$, $A = B = q^{1/2}\pi^{-1/2}$, $\alpha_i(p) = \chi(p)$, $\beta_i(p) = \bar{\chi}(p)$ and $c = 0$. In this case there is no restriction in supposing the test function F to be even. We rewrite Theorem 2.1 in this case:

THEOREM 2.2. *Notations as in Theorem 2.1, the following formula holds:*

$$\sum_{\rho}\Phi(\rho) - \delta_{\chi_0}\big(\Phi(0)+\Phi(1)\big) + I(\frac{1}{2},\frac{\mathfrak{a}}{2}) + J(\frac{1}{2},\frac{\mathfrak{a}}{2}) =$$

$$F(0)\log\frac{q}{\pi} - \sum_{n\geq 1}\big(\chi(n)+\bar{\chi}(n)\big)\frac{\Lambda(n)}{\sqrt{n}}F(\log n)$$

where ρ runs over the zeros of $L(s,\chi)$ inside the strip $0 \leq \Re s \leq 1$ and $\delta_{\chi_0} = 1$ if $\chi = \chi_0$ (in which case $L(s,\chi) = \zeta(s)$) and zero otherwise.

2.2.2. Explicit formula for $L(s,f)$. If f is a newform, we showed in section 1.2.5 it has an Euler product expansion. From the functional equation (Theorem 1.5) we see that the hypothesis of Theorem 2.1 are satisfied by taking $\Lambda_1(s) =$

$\Lambda_2(s) = \Lambda(s+1/2,f)$, so that $A = \sqrt{N}/2\pi$, $M = 1$, $M' = 2$, $a_j = 1$, $b_j = 0$, $\alpha_1(p) = \beta_1(p) = \alpha_p/\sqrt{p}$, $\alpha_2(p) = \beta_2(p) = \bar{\alpha}_p/\sqrt{p}$ if $p \nmid N$ (notations as in section 1.2.5), whereas $\alpha_1(p) = \beta_1(p) = a_p/\sqrt{p}$ and $\alpha_2(p) = \beta_2(p) = 0$ if $p|N$. We rewrite Theorem 2.1 in this case. It is useful for this purpose to introduce numbers c_n indexed on prime powers by

$$c_{p^m} = \begin{cases} a_p^m & \text{if } p \text{ divides } N, \\ \alpha_p^m + \bar{\alpha}_p^m & \text{otherwise.} \end{cases}$$

REMARKS. (1) It follows from this and remark 1 of section 1.2.5 that $c_p = a_p = \hat{f}(p)$ for every prime p. Also we have the bound

$$|c_n(f)| \leq 2\sqrt{n} \tag{2.3}$$

valid for all n.

(2) The c_n's occur in the coefficients of the Dirichlet series

$$-\frac{L'}{L}(s,f) = \sum_{n=1}^{\infty} \frac{c_n(f)}{n^s}\Lambda(n)$$

THEOREM 2.3 (Weil's explicit formula). *Let $F\colon \mathbb{R} \to \mathbb{R}$ be an even function satisfying:*

- *there is an $\epsilon > 0$ such that $F(x)\exp((1+\epsilon)x)$ is integrable and of bounded variation,*
- *the function $(F(x) - F(0))/x$ is of bounded variation.*

Let $\phi = \hat{F}$ be the Fourier transform of F.

Then

$$\sum_{L(1+i\gamma,f)=0} \phi(\gamma) = F(0)\log N - 2F(0)\log(2\pi)$$
$$+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Gamma'}{\Gamma}(1+it)\phi(t)dt - 2\sum_{n=1}^{\infty}\frac{c_n(f)}{n}\Lambda(n)F(\log n)$$

where the sum on the left hand side is over the zeroes of $L(s,f)$ in the critical strip $1/2 \le \Re(1+i\gamma) \le 3/2$.

2.3. Orthogonality and quasi-orthogonality

2.3.1. Orthogonality of characters. Let $q \ge 1$ be an integer. The following two formulas express an "orthogonality" property in the group $\hat{G}$ of characters modulo q. Note that $|\hat{G}| = \varphi(q)$.

THEOREM 2.4. *We have*

$$\sum_{\chi\in\hat{G}} \chi(n) = \begin{cases} \varphi(q) & \text{if } n \equiv 1 \pmod q \\ 0 & \text{otherwise} \end{cases}$$

and

$$\sum_{n\in G} \chi(n) = \begin{cases} \varphi(q) & \text{if } \chi = \chi_0 \\ 0 & \text{otherwise} \end{cases}$$

PROOF. We prove the first formula, the second is similar (dual proof). Note that the formula is trivial if $n \equiv 1 \pmod q$. In the remaining case, there exists $\chi' \in \hat{G}$ such that $\chi'(n) \neq 1$ (otherwise $\hat{G}$ would also be the character group of a smaller group). Hence

$$\sum_{\chi\in\hat{G}} \chi(n) = \sum_{\chi\in\hat{G}} \chi\chi'(n) = \chi'(n)\sum_{\chi\in\hat{G}} \chi(n)$$

implying the result. □

It was this property which played a crucial role in Dirichlet's proof of the infinity of primes in an arithmetic progression.

2.3.2. Quasi-orthogonality of Fourier coefficients of cusp forms. The situation in the case of $S_2(N)$ is similar yet there are some important differences. We can only produce an approximate orthogonality, called quasi-orthogonality. Using the Poincaré series which give an explicit basis of $S_2(N)$, Petersson was able to prove the following theorem (cf. [**Iwa**$_1$]).

THEOREM 2.5. *Write*

$$f(z) = \sum_{n=1}^{\infty} \hat{f}(n) q^n$$

for the q-expansion of a form in $S_2(N)$. *Let* $\{f_1, \ldots, f_r\}$ *be an orthogonal basis of* $S_2(N)$. *Then*

$$\sum_{1 \le i \le r} \frac{\hat{f}_i(n)\hat{f}_i(m)}{4\pi \langle f_i, f_i \rangle \sqrt{n}\sqrt{m}} = \delta_{mn} - 2\pi \sum_{c \equiv 0 (\bmod N)} J_1\left(\frac{4\pi\sqrt{mn}}{c}\right) \frac{S(m,n,c)}{c}$$

where $\delta_{mn} = 0$ *unless* $m = n$ *in which case it is* 1, J_1 *is the Bessel function of order* 1 *and* $S(m,n,c)$ *is the Kloosterman sum*

$$S(m,n,c) = \sum_{\substack{d (\bmod c) \\ d\bar{d} \equiv 1 (\bmod c)}} \exp\left(2\pi i \frac{md + n\bar{d}}{c}\right).$$

Using the estimate $|J_1(x)| \le x/2$ when $x \ge 0$ and Weil's estimate

$$|S(m,n,c)| \le (m,n,c)^{1/2} d(c) c^{1/2}$$

it is a simple matter to deduce

COROLLARY 2.1. *Let* $\sum'$ *denote the sum over an orthogonal basis of* $S_2(N)$. *Then*

$$\sum_f{}' \frac{\hat{f}(m)\hat{f}(n)}{4\pi\langle f, f\rangle\sqrt{m}\sqrt{n}} = \delta_{mn} + O(N^{-3/2}(m,n)^{1/2}\sqrt{mn}).$$

The preceding formula (with $m = 1$) is the starting point to prove the fundamental Lemma 4.2, by introducing the L-function of the symmetric square of f evaluated at 2.

2.3.3. The L-function of the symmetric square of f. Although we will not use it, we sometimes make assumptions on a special L-function, denoted $L(s, \mathrm{sym}^2 f)$ or $L(s, \vee^2 f)$, where $f \in S_2(N)^{\mathrm{new}}$. The purpose of this function is to relate $\langle f, f\rangle$ to the value $L(2, \vee^2 f)$. Using an estimate on $L(2, \vee^2 f)$ obtained by analytic means, it is then possible to estimate effectively $\sum'_f \hat{f}(n)$ as Murty did in his proof of Lemma 4.2. Let us recall here the basic facts about $L(s, \vee^2 f)$. This exposition follows [**Shi**$_2$]. Let f be a newform of weight 2 and level N. Each Euler factor of $L(s, f)$ can be factored as

$$1 - \hat{f}(p)p^{-s} + \chi(p)p^{1-2s} = (1 - \alpha_p p^{-s})(1 - \beta_p p^{-s})$$

where $\chi(p) = 0$ if $p \mid N$ (cf. section 1.2.5).

DEFINITION 2.1. The following Euler product converges if $\Re s > 2$ and defines the L-function of the symmetric square of f.

$$L(s, \vee^2 f) = \prod_p [(1 - \alpha_p^2 p^{-s})(1 - \alpha_p\beta_p p^{-s})(1 - \beta_p^2 p^{-s})]^{-1}.$$

It can be easily seen (cf. [**C-S**, p. 124]) that

$$L(s, \vee^2 f) = \zeta_N(2s-2)\sum_{n\geq 1}\frac{\hat{f}(n^2)}{n^s},$$

where ζ_N is the Riemann zeta function with the Euler factors corresponding to $p \mid N$ removed. The following theorem gives the analytic continuation of $L(s, \vee^2 f)$.

THEOREM 2.6 (Shimura). *Define*

$$R(s) = \pi^{-3s/2} N^s \left(\Gamma\left(\frac{s}{2}\right)\right)^2 \Gamma\left(\frac{s+1}{2}\right) L(s, \vee^2 f).$$

Then $R(s)$ can be continued to a meromorphic function on the whole plane which is holomorphic except possibly for simple poles at $s = 2$ and $s = 1$. Furthermore $R(s)$ satisfies the functional equation

$$R(3 - s) = R(s).$$

The following theorem characterises the special value at 2.

THEOREM 2.7. *We have*

$$\frac{N}{12\zeta(2)} L(2, \vee^2 f) = 4\pi \langle f, f \rangle + O_\epsilon(N^{1/2+\epsilon}),$$

provided $f \in S_2(N)^{new}$ (when N is supported on finitely many primes, the O term is $O(\log N)$).

2.4. Main conjectures

We introduce here the main conjectures in this field.

CONJECTURE 1 (Riemann hypothesis). *Let $L(s)$ be an $\mathcal{S}$-function , that is a function of the $\mathcal{S}$ family defined by Selberg (cf. [**Mur**$_2$]) (for example $L(s, \chi)$ or $L(s + 1/2, f)$, but also $\zeta_k(s)$, $L(s + 1, \vee^2 f)$, etc.). Then its zeros lie on the line $\Re s = 1/2$, except "trivial" ones.*

Nothing of the sort is known, unfortunately[1]. For the simplest of all L-functions, namely Riemann's zeta-function, it is known that the zeros (except of course the simple trivial ones at negative even integers arising out of the functional equation) lie in $0 < \Re s < 1$ and this is equivalent to the prime number theorem (Hadamard and de la Vallée Poussin), but no $\epsilon > 0$ is known for which they lie in $\epsilon < \Re s < 1-\epsilon$. The shape of zero-free regions are of the kind given by the following result.

THEOREM 2.8 (Vinogradov, Korobov (1958)).
Write $t = \Im s$. Then $\zeta(s) \neq 0$ in the region

$$\Re s \geq 1 - \frac{c(\alpha)}{\log^{\alpha}(|t|+2)}$$

for any $\alpha > 2/3$.

This hypothesis is essential to get the full power of the explicit formula. In fact, a correct use of the explicit formulas for the same purpose without assuming the Riemann hypothesis (RH) for $L(s,f)$ or $L(s,\chi)$ was achieved only recently in the work of Kowalski and Michel [**K-M**$_1$]. However, in their case, the unconditional bound they find is much worse than ours. Since in this work we are interested primarily in understanding the phenomenon, we may as well assume the RH for specific L-functions. Our goal should ultimately be the following.

CONJECTURE 2 (Brumer-Murty). *We have*

$$\lim_{N\to\infty} \frac{\operatorname{rank} J_0(N)}{\dim S_2(N)} = \frac{1}{2}$$

REMARK. Actually in this work we implicitly assume the general Birch and Swinnerton-Dyer conjecture which predicts among other facts that the algebraic

[1] There is however great numerical evidence, cf. [**Odl**].

rank of $J_0(N)$ should equal the order of vanishing at 1 of $L(s, J_0(N))$ (called the analytic rank of $J_0(N)$). Thus henceforth we will use the word rank, meaning thereby the analytic rank.

Brumer finds evidence for this conjecture. Murty also points out the lower limit is true. The conjecture means that on the average the $L(s, f)$ will vanish at 1 with the lowest admissible order (1 if $f \in S_2^+(N)$, 0 otherwise). This conjecture agrees with the following conjecture of Greenberg which was the motivation for Murty's work and his formulation of the Brumer-Murty conjecture.

Conjecture 3 (Greenberg). *Let Σ be a finite set of primes and N an integer supported on Σ. Then, except for at most finitely many f's,* $\operatorname{ord}_{s=1} L(s, f) = 0$ *or* 1 *for any newform $f \in S_2(N)^{new}$.*

There is an analog for $L(s, \chi)$. In this case, no parity argument is available. It is commonly believed therefore that

Conjecture 4 (Stronger hypothesis). *If χ is a Dirichlet character then*

$$L(\frac{1}{2}, \chi) \neq 0.$$

A weaker version would be

Conjecture 5 (Weaker hypothesis). *Let $r_\chi = \operatorname{ord}_{s=1/2} L(s, \chi)$. Then*

$$\sum_{\chi \bmod q} r_\chi = o\big(\varphi(q)\big)$$

where summation is over all characters modulo q.

Unfortunately, both hypothesis are not yet proven [2], as is the Brumer-Murty conjecture. We will present Murty's conditional (RH) approach to this problem in the following chapter.

In the subsequent chapters we will see how the same method can be applied to attack the Brumer-Murty conjecture and we will show that assuming the RH for $L(s, f)$ Greenberg's conjecture holds on average, for prime level N.

[2] Cf. however next chapter for partial results.

CHAPTER 3

The non-vanishing of $L(\frac{1}{2},\chi)$

Let χ be a Dirichlet character to the modulus q. As mentioned in chapter 2, none of the $L(1/2,\chi)$ should vanish; unfortunately, there are only partial results in this direction. For example, Balasubramanian and Kumar Murty [**B-M**] proved that a positive proportion of $L(1/2,\chi)$ doesn't vanish, when q is prime (cf also the work of Soundararajan [**Sou**]). Recently, Iwaniec and Sarnak [**I-S**$_1$] proved the same with an explicit proportion of $\frac{1}{3}$. The best lower bound (conditional on the RH for $L(s,\chi)$) for this proportion belongs to Ram Murty [**Mur**$_3$], who uses the explicit formula method to get a proportion of 50%.

We will review here his method and generalise it to composite q, in order to understand a similar situation in the case of cusp forms. For a motivation of our choices (test function, etc.) cf. next chapter. The starting point is Theorem 2.2. We note that if we define $\phi(t)=\Phi(1/2+it)$ then ϕ is the Fourier transform of F.

Theorem 3.1. *Assume the RH for all $L(s,\chi)$. Let $r_\chi=\operatorname{ord}_{s=1/2}L(s,\chi)$, then*

$$\limsup_{q\to\infty}\frac{\displaystyle\sum_{\chi\,(\bmod q)} r_\chi}{\varphi(q)}\le\frac{1}{2}$$

where φ is Euler's totient function.

Remark. R. Murty proves this only when q is prime.

Corollary 3.1. *Asymptotically, at least 50% of $L(1/2,\chi)$ will not vanish.*

PROOF. Because we are taking F to be even and we are assuming RH for $L(s,\chi)$, in order to find an upper bound on r_χ we need to apply Theorem 2.2 to functions F so that the corresponding ϕ be non-negative on the real line. Let F_x be the function

$$F_x = \max\left(1 - \left|\frac{u}{\log x}\right|, 0\right).$$

Then the corresponding Fourier transform $(\log x)\phi(t\log x)$ is the entire function

$$\frac{4}{\log x}\left(\frac{\sin\frac{t\log x}{2}}{t}\right)^2.$$

The following lemma will deal with the archimedean part of the explicit formula (I and J terms).

LEMMA 3.1. *We have*

$$I(\frac{1}{2},\frac{\mathfrak{a}}{2}) = J(\frac{1}{2},\frac{\mathfrak{a}}{2}) = \frac{\log x}{2\pi}\int_{-\infty}^{+\infty}\phi(t\log x)\frac{\Gamma'}{\Gamma}\left(\frac{1+2\mathfrak{a}}{4} + i\frac{t}{2}\right)dt \ll 1,$$

as $x \to \infty$.

PROOF OF LEMMA. The first equality follows from the parity of ϕ. Let us prove the final bound. For this we need the asymptotic formula

$$\frac{\Gamma'}{\Gamma}(s) = \log s + O\left(\frac{1}{|s|}\right) = \log|s| + \arg(s) + O\left(\frac{1}{|s|}\right)$$

valid for $-\pi+\delta < \arg s < \pi - \delta$. Then decompose the integral into the three parts corresponding to $A_1(t) = \log|s|$, $A_2(t) = \arg(s)$ and $A_3(t) = O(|s|^{-1})$ and change variables to get for each integral the expression

$$\int_{-\infty}^{+\infty}\phi(t)A_k\left(\frac{t}{\log x}\right)dt.$$

Note that for $x > e$ we have $|A_1(t/\log x)| \ll |A_1(t)|$, hence the first integral is $O(1)$. Also as t varies in $\mathbb{R}$, $A_2(t/\log x)$ and $A_3(t/\log x)$ are bounded by some absolute constant, so the lemma follows. □

Now we apply Theorem 2.2 with the above function to $L(s,\chi')$, where χ' is the primitive character of conductor $q'|q$ which induces χ and we sum over $\chi \pmod q$. Note that

$$(\log x)\,\phi\left(\frac{i\log x}{2}\right) = \frac{16}{\log x}\left(\frac{\sin\frac{i\log x}{4}}{i}\right)^2 = \frac{16}{\log x}\sinh^2\left(\frac{\log x}{4}\right) = \frac{4\sqrt{x}}{\log x} + O(\frac{1}{\log x}).$$

Hence we have

$$\log x \sum_{\chi \,(\bmod q)} r_\chi \le \varphi(q)\log q + O(\varphi(q)) + 4\sqrt{x} - \sum_{n\le x}\sum_{\chi\,(\bmod q)} \big(\chi'(n)+\bar{\chi}'(n)\big)\frac{\Lambda(n)}{\sqrt{n}}\left(1-\frac{\log n}{\log x}\right). \tag{3.1}$$

We must deal with the summation on the right-hand side. Note that $\chi(n) = \chi'(n)$ when $(n,q)=1$. Hence we can replace

$$\sum_{n\le x}\sum_{\chi\,(\bmod q)} \big(\chi'(n)+\bar{\chi}'(n)\big)\frac{\Lambda(n)}{\sqrt{n}}\left(1-\frac{\log n}{\log x}\right)$$

by

$$\sum_{n\le x}\sum_{\chi\,(\bmod q)} \big(\chi(n)+\bar{\chi}(n)\big)\frac{\Lambda(n)}{\sqrt{n}}\left(1-\frac{\log n}{\log x}\right) + A(x)$$

where

$$\begin{aligned} A(x) &\le 2\varphi(q)\sum_{\substack{n\le x\\(n,q)>1}} \frac{\Lambda(n)}{\sqrt{n}}\left(1-\frac{\log n}{\log x}\right) \\ &\le 2\varphi(q)\sum_{k\le\frac{\log x}{\log q}} \frac{\log q}{q^{k/2}} \\ &\le 2\varphi(q)\frac{\log q}{\sqrt{q}-1} \ll \varphi(q). \end{aligned}$$

Now applying orthogonality relations we deduce

$$\sum_{n\le x}\sum_{\chi(\bmod q)}\big(\chi(n)+\bar{\chi}(n)\big)\frac{\Lambda(n)}{\sqrt{n}}\left(1-\frac{\log n}{\log x}\right)$$
$$=2\varphi(q)\sum_{\substack{n\le x\\ n\equiv 1(\bmod q)}}\frac{\Lambda(n)}{\sqrt{n}}\left(1-\frac{\log n}{\log x}\right)\ge 0$$

hence we can discard it in (3.1). After setting $x=(\varphi(q))^2$, in view of (1.2) we get the bound

$$\sum_{\chi(\bmod q)} r_\chi \le \frac{1}{2}\varphi(q)+O\left(\frac{\varphi(q)}{\log x}\right)$$

which proves the theorem. □

CHAPTER 4

A bound for the analytic rank of $J_0(N)$

4.1. Trivial bounds

We use the explicit formula in the form of Theorem 2.3 to get the following.

THEOREM 4.1. *We have the following unconditional bound*

$$\operatorname{rank}\big(J_0(N)\big) \ll \dim S_2(N) \log N$$

PROOF. Let r_f be the order of vanishing of $L(s, f)$ at $s = 1$. From the definition of analytic rank we see that we need to bound r_f by $O(\log N)$. To do this, it suffices to choose an even test function F so that

$$\Re\Phi(s) \geq 0 \quad \text{for all } s \text{ satisfying } 0 \leq \Re s \leq 1. \tag{H}$$

This is given by the following.

LEMMA 4.1. *Let G be an even test function satisfying the conditions of Theorem 2.3. If $\int_{-\infty}^{\infty} G(x)e^{itx}\,dx \geq 0$ for all $t \in \mathbb{R}$, then the function F defined by $F(x) = G(x)/\cosh(x/2)$ satisfies hypothesis* (H).

PROOF OF LEMMA. Write $s = \sigma + it$ with $t, \sigma \in \mathbb{R}$. We have

$$\begin{aligned}\Re\Phi(s) &= \int_{-\infty}^{+\infty} F(x)e^{(\sigma-1/2)x}\cos(tx)\,dx \\ &= \int_{-\infty}^{+\infty} F(x)\cosh\{(\sigma - \frac{1}{2})x\}\cos(tx)\,dx.\end{aligned}$$

By the Phragmen-Lindelöf theorem (maximum principle inside a vertical strip), since

$$\Re\Phi(s) \leq \int_{-\infty}^{+\infty} F(x) \cosh \frac{x}{2} \, dx < \infty \quad \text{for all } 0 \leq \Re s \leq 1,$$

$\Re\Phi(s)$ will be non-negative if and only if

$$\int_{-\infty}^{+\infty} F(x) \cosh \frac{x}{2} \cos(tx) \, dx \geq 0 \quad \text{for all } t \in \mathbb{R}$$

thus showing the lemma. □

Then we take F as in the lemma. For example we can take

$$F(x) = \max\left(\frac{1 - |x|}{\cosh(x/2)}, 0\right).$$

Applying Theorem 2.3 and discarding all the zeros except 1/2, we obtain

$$r_f \Phi(\frac{1}{2}) \ll \log N + O(1) + 4 \sum_n \frac{F(\log n)}{\sqrt{n}} \Lambda(n)$$

in view of bound (2.3), thus proving the theorem. □

4.2. Murty's bound

We review an argument of Murty [**Mur**$_1$] to get an upper bound for the analytic rank of $J_0(N)$, N prime. In the next section we modify the method to deal with composite N.

We apply Theorem 2.3 to the function $F_\lambda(u) = F(u/\lambda)$.

Remark. It is natural to use F_λ because it gives, as λ grows, an approximation in $\mathcal{D}'(\mathbb{R})$ to the distribution 1. The Fourier transform is a continuous transformation hence $\hat{F}_\lambda$ will converge in $\mathcal{D}'$ to the Fourier transform of 1 which is δ_0, the Dirac delta at 0. This is why this family of functions is a good choice to single out the contribution of the central zero.

Note that $\hat{F}_\lambda(t) = \lambda\phi(\lambda t)$. Assuming the RH for $L(s,f)$ and $\phi(t) \geq 0\ \forall t \in \mathbb{R}$, we have that, after writing $\lambda = \log x$

$$r_f \phi(0) \log x \leq F(0) \log N - 2\sum_{n=1}^{\infty} \frac{c_n(f)}{n} \Lambda(n) F(\frac{\log n}{\log x}) + O(1)$$

where $r_f = \operatorname{ord}_{s=1} L(s,f)$. Assume further that F is continuous and supported in $(-1,1)$ and let S be a collection of forms in $S_2(N)$, then we get

$$\sum_{f\in S} r_f \phi(0) \log x \leq |S|\, F(0) \log N - 2\sum_{n\leq x}(\sum_{f\in S} c_n(f))\frac{\Lambda(n)}{n} F(\frac{\log n}{\log x}) + O(|S|)$$

whence the fundamental estimate

$$\begin{aligned} \sum_{f\in S} r_f &\leq |S| \frac{F(0)}{\phi(0)} \frac{\log N}{\log x} \\ &\quad - \frac{2}{\phi(0)\log x}\sum_{n\leq x}(\sum_{f\in S} c_n(f))\frac{\Lambda(n)}{n} F(\frac{\log n}{\log x}) + O(\frac{|S|}{\log x}). \end{aligned} \tag{4.1}$$

In the case of prime conductor N we have that an orthogonal basis for $S_2(N)$ is given by $S = \{$normalised newforms of level $N\}$ and (4.1) becomes

$$\begin{aligned} {\sum}' r_f &\leq \dim S_2(N) \frac{F(0)}{\phi(0)} \frac{\log N}{\log x} \\ &\quad - \frac{2}{\phi(0)\log x}\sum_{n\leq x}({\sum}' c_n(f))\frac{\Lambda(n)}{n} F(\frac{\log n}{\log x}) + O(\frac{N}{\log x}) \end{aligned}$$

where ${\sum}'$ denotes the sum over newforms. We now proceed as in [**Mur**$_1$] to estimate the average analytic rank. Decompose the double sum into

$$\begin{aligned} &\sum_{n\leq x}({\sum}' c_n(f))\frac{\Lambda(n)}{n} F(\frac{\log n}{\log x}) \\ &= \sum_{p\leq x} \cdots + \sum_{p^2\leq x} \cdots + \sum_{\substack{a\geq 3 \\ p^a\leq x}} \cdots \\ &= {\sum}_1 + {\sum}_2 + {\sum}_3 . \end{aligned}$$

Estimating $\sum_3$ trivially get

$$\sum\nolimits_3 \ll N \sum_{\substack{p^a \\ a \geq 3}} \frac{\log p}{p^{a/2}} \ll N.$$

The bound on $\sum_1$ is a consequence of Murty's trace formula:

LEMMA 4.2 (Murty). *Assuming the Lindelöf hypothesis for $L(s, \vee^2 f)$, let $\theta > 0$ (unconditionally, $\theta = 1/2$). Then, for any $T > 0$ we have*

$$\sum\nolimits' \hat{f}(n) = \delta(n) \frac{N^2}{12\,\varphi(N)} \Big\{ \prod_{p|N} (1 - p^{-2}) + O(T^{-1/2+\epsilon} n^{1/4}) \Big\}$$

$$+ O(N^{-1/2} n T d(n) + \sqrt{n} d(n) N^{1+\theta} T^{-1/2})$$

where $d(n) = \sum_{d|n} 1$, $\delta(n) = 1$ if n is a square and zero otherwise and φ is Euler's totient function.

REMARK. This looks like a heavy hypothesis to get an estimate for the trace of $\mathbf{T}_n$, though it follows from the RH for the same function. Murty stated the theorem only for prime N (with minor typographical errors, as was already pointed out by Akbary [**Akb**], who derived a similar trace formula for $S_2(N)$).

Using this lemma we get

$$\sum\nolimits_1 \ll \frac{Tx}{\sqrt{N} \log x} + N^{1+\theta} \frac{x^{1/2}}{T^{1/2} \log x}$$

Next since $c_{p^2}(f) = \hat{f}(p^2) - p$ we remark that $\sum_2$ splits into two subsums: the one corresponding to the modular coefficients is again easily dealt with by the trace lemma. We find the bound

$$O(\frac{T\sqrt{x}}{\sqrt{N} \log x} + N^{1+\theta} \frac{x^{1/4}}{T^{1/2} \log x} + N).$$

It remains to evaluate the part with p, which is given in the following lemma.

LEMMA 4.3. *Let $F\colon \mathbb{R} \to \mathbb{R}$ be a continuous even function with compact support. Then*

$$\lim_{x\to\infty} \frac{1}{\log x} \sum_p \frac{\log p}{p} F(\frac{2\log p}{\log x}) = \frac{\phi(0)}{4}$$

This is a consequence (by uniform approximation) of the following result.

LEMMA 4.4. *Take F as before such that also $F \in C^1(\mathbb{R})$, with $F' \in L^1(\mathbb{R})$ and $F(u) = o(1/u)$. Then*

$$\sum_p \frac{\log p}{p} F(\frac{2\log p}{\log x}) = \frac{\phi(0)}{4} \log x + O(\|F'\|_1)$$

PROOF OF LEMMA 4.4. Indeed by partial summation

$$\sum_{p\le y} \frac{\log p}{p} F(\frac{2\log p}{\log x}) = (\sum_{p\le y} \frac{\log p}{p}) F(\frac{2\log y}{\log x}) - \int_1^y (\sum_{p\le t} \frac{\log p}{p}) F'(\frac{2\log t}{\log x}) \frac{2}{t\log x}\, dt.$$

As we let $y \to \infty$ we get

$$\begin{aligned}
\sum_p \frac{\log p}{p} F(\frac{2\log p}{\log x}) &= -\int_1^\infty (\sum_{p\le t} \frac{\log p}{p}) F'(\frac{2\log t}{\log x}) \frac{2}{t\log x}\, dt \\
&= -\int_1^\infty (\log t) F'(\frac{2\log t}{\log x}) \frac{2}{t\log x}\, dt + O(\|F'\|_1) \\
&= -\int_0^\infty uF'(\frac{2u}{\log x}) \frac{2}{\log x}\, du + O(\|F'\|_1) \\
&= -\big[uF(\frac{2u}{\log x})\big]_0^\infty + \int_0^\infty F(\frac{2u}{\log x})\, du + O(\|F'\|_1) \\
&= \frac{\log x}{2} \int_0^\infty F(w)\, dw + O(\|F'\|_1) \\
&= \frac{\log x}{4} \phi(0) + O(\|F'\|_1).
\end{aligned}$$

□

Fitting the analysis of the previous lemma into $\sum_2$, we find that

$$\frac{\sum' r_f}{\dim S_2(N)} \leq \frac{F(0)}{\phi(0)}\frac{\log N}{\log x} + \frac{1}{2} + O(\frac{Tx}{(\log^2 x)N^{3/2}} + N^\theta \frac{x^{1/2}}{T^{1/2}\log x} + \frac{1}{\log x}).$$

This result can be made more precise under the following conjecture (cf. [**Sel**$_1$]).

CONJECTURE 6 (Linnik-Selberg). *Let $S(m,n,c)$ be the Kloosterman sum*

$$S(m,n,c) = \sum_{\substack{(d,c)=1 \\ d (\bmod c)}} \exp(2\pi i \frac{md + n\bar{d}}{c})$$

where $d\bar{d} \equiv 1 \bmod c$. Then we have

$$\sum_{c \leq x} \frac{S(m,n,c)}{c} \ll x^\epsilon$$

when $x > (m,n)^{1/2+\epsilon}$.

REMARK. We take the opportunity to point out a misprint in [**Mur**$_1$, p. 267], where an extra power of 1/2 slipped into the conjecture.

We would like to stress that there are partial steps towards this conjecture. The "trivial" bound

$$\sum_{c \leq x} \frac{S(m,n,c)}{c} \ll x^{1/2}$$

was improved first by Kuznietsov [**Kuz**] in the case of the full modular group to

$$\sum_{c \leq x} \frac{S(m,n,c)}{c} \ll x^{1/6}(\log x)^{2/3}.$$

The most general result (for any congruence subgroup) is held by Luo, Rudnick and Sarnak [**L-R-S**] who prove

$$\sum_{\substack{c \leq x \\ c \equiv 0 (\bmod N)}} \frac{S(m,n,c)}{c} \ll x^{2/5}.$$

Assuming this conjecture, $N^{3/2}$ can be replaced by N^2, thus allowing to take $T = x^{1+\eta}$ and $x = N^{1-\epsilon}$ to get

$$\limsup_{N\to\infty} \frac{\sum' r_f}{\dim S_2(N)} \leq \frac{F(0)}{\phi(0)} + \frac{1}{2} \tag{4.2}$$

By choosing $F(t) = \max(1 - |t|, 0)$ which is the square convolution of the characteristic function of the interval $(-1/2, 1/2)$, we have $\phi(t) = 4(\sin(t/2)/t)^2$ (with $\phi(0) = 1$) and get

THEOREM 4.2 (Murty). *We have, under the RH for all $L(s, f)$ and the Linnik-Selberg conjecture,*

$$\limsup_{\substack{N\to\infty \\ N \text{ prime}}} \frac{\sum' r_f}{\dim S_2(N)} \leq \frac{3}{2}.$$

4.3. Generalisation to composite N

In this section we explain how the previous method can be generalized to any level N in order to obtain the same bound. The following theorem gives the structure of the vector space $S_2(N)$ as a direct sum of eigenspaces for all Hecke operators $\mathbf{T}_n$ with $(n, N) = 1$ (cf. [**A-L**] or [**DDT**, p. 23]).

THEOREM 4.3. *Let $M \mid N$, $f \in S_2(M)^{\text{new}}$ and $V_{f,M} =< f(\sigma z) >_{\sigma|(N/M)}$. Then*

$$S_2(N) = \bigoplus_{M|N} \bigoplus_{f\in S_2(M)^{\text{new}}} V_{f,M}$$

where the spaces $V_{f,M}$ are orthogonal with respect to the Petersson inner product and correspond to the eigenspace decomposition of $S_2(N)$ with respect to a system of eigenvalues for the family $\mathbf{T}_n$ with $(n, N) = 1$.

From this theorem we retrieve the factorisation of $L(s, J_0(N))$.

THEOREM 4.4. *We have the following decomposition:*

$$L(s, J_0(N)) = \prod_{M|N} \prod_{f\in S_2(M)^{\text{new}}} \prod_{\sigma|(N/M)} L(s, f(\sigma z))$$

Furthermore, $\operatorname{ord}_{s=1} L(s, f(\sigma z)) = \operatorname{ord}_{s=1} L(s, f(z)) = r_f$.

COROLLARY 4.1. *The analytic rank of $J_0(N)$ is equal to*

$$\sum_{M|N} \sum_{f\in S_2(M)^{\text{new}}} d(\frac{N}{M})\, r_f$$

and for $g \in V_{f,M}$ we have $r_g \geq r_f$.

Now for each M, f as before denote by $\mathcal{G}_{f,M}$ an orthogonal basis of $V_{f,M}$ and define $S = \bigcup_{M,f} \mathcal{G}_{f,M}$. Then S is an orthogonal basis of $S_2(N)$. From Theorem 4.4 and (4.1) we deduce

$$\begin{aligned}\text{Analytic rank } J_0(N) \leq \dim S_2(N) \frac{F(0)}{\phi(0)} \frac{\log N}{\log x} \\ - \frac{2}{\phi(0)\log x} \sum_{n\leq x} \Big(\sum_{\substack{M|N \\ f\in S_2(M)^{\text{new}}}} \sum_{g\in\mathcal{G}_{f,M}} c_n(f)\Big) \frac{\Lambda(n)}{n} F(\frac{\log n}{\log x}) + O(\frac{N}{\log x}).\end{aligned}$$

Since by definition $\hat{f}(n) = \hat{g}(n)$ for any $g \in \mathcal{G}_{f,M}$ and $(n, N) = 1$, whence $c_n(f) = c_n(g)$, we can rewrite the sum over n as

$$\begin{aligned}&\sum_{n\leq x} \Big(\sum_{\substack{M|N \\ f\in S_2(M)^{\text{new}}}} \sum_{g\in\mathcal{G}_{f,M}} c_n(f)\Big) \frac{\Lambda(n)}{n} F(\frac{\log n}{\log x}) \\ &= \sum_{\substack{p\leq x \\ (p,N)=1}} \cdots + \sum_{\substack{p^2\leq x \\ (p,N)=1}} \cdots + \sum_{\substack{a\geq 3,\, p^a\leq x \\ (p,N)=1}} \cdots + \sum_{\substack{n\leq x \\ (n,N)>1}} \cdots \\ &= \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3 + \sum\nolimits_4\end{aligned}$$

Now in $\sum_i$, $(i = 1, 2, 3)$ write

$$\sum_{\substack{M|N \\ f\in S_2(M)^{\rm new}}} \sum_{g\in\mathcal{G}_{f,M}} c_n(f) = \sum_{f\in S} c_n(g)$$

Then since S is an orthogonal basis the trace lemma 4.2 applies and we can estimate $\sum_i$, $(i = 1, 2, 3)$ in the same way as before (note that in lemma 4.4 the same conclusion holds if we sum over p not dividing N). Notice that $N^2/\varphi(N) \ll N \log\log N$, hence we get the same estimates as in the prime case with an extra factor in the error terms of $\log\log N$ which is negligible.

Finally, to deal with $\sum_4$ it suffices to notice that

$$\sum\nolimits_4 \ll N \sum_{p^a,\ p|N} \frac{\log p}{p^{a/2}} \ll N \sum_{p|N} \frac{\log p}{\sqrt{p}} \ll N \log^{1/2} N$$

We summarize what we proved in this section

THEOREM 4.5. *Under the RH for all $L(s, f)$ and the Linnik-Selberg conjecture we have*

$$\limsup_{N\to\infty} \frac{\sum' r_f}{\dim S_2(N)} \leq \frac{3}{2}$$

REMARK. Actually Brumer [**Bru**] practically proves this because he finds the same bound for $\operatorname{rank}\big(J_0^{\rm new}(N)\big)$. Since he makes use of the Selberg trace formula, he can avoid to invoke the Linnik-Selberg conjecture to get the same result. We think that the same should be true also using the Petersson approach.

4.4. Using other test functions

In the light of the preceding sections, we see that we can sharpen the bound (4.2) if we can find an even F, continuous, supported in $(-1, 1)$, with non negative Fourier transform ϕ (on the real line) such that $F(0)/\phi(0) < 1$ (possibly $< \epsilon$).

Unfortunately this is impossible because of Krein's theorem on entire functions (cf. [**Lev**, p. 437]).

DEFINITION 4.1. An entire function is said to be of exponential type if

$$a = \sup \left\{ b \in \mathbb{R} : \frac{|\phi(z)|}{e^{b|z|}} \ll 1 \ \forall z \in \mathbb{C} \right\} < \infty.$$

a is then called the type of the function.

Recall that we have the following characterisation of the Fourier transform of L^2-functions with compact support (cf. [**Kat**, p. 176]).

THEOREM 4.6 (Paley-Wiener). *A necessary and sufficient condition for a function $F \in L^2(\mathbb{R})$ to have compact support in $(-a, a)$ is that its Fourier transform be in $L^2(\mathbb{R})$ and entire of exponential type at most a.*

THEOREM 4.7 (Krein). *Let ϕ be an entire function satisfying:*

- $\phi(t) \geq 0, \ \forall t \in \mathbb{R}$,
- *ϕ is entire of exponential type a,*
- *ϕ is bounded on $\mathbb{R}$.*

Then there exists an entire function ψ of exponential type $a/2$ such that $\phi(t) = |\psi(t)|^2, \ \forall t \in \mathbb{R}$.

PROOF OF CLAIM. Notice that $\hat{F}$ always satisfies the second (with $a = 1$) and third hypothesis of the theorem by the Paley-Wiener theorem and the Riemann-Lebesgue lemma, hence if we assume the first hypothesis we can introduce the corresponding ψ. Also let $G = 2\pi\hat{\psi}$. Then since ψ has type $1/2$ and has finite L^2 norm, we get by the Paley-Wiener theorem that G is supported in $(-1/2, 1/2)$. Moreover, if $\Gamma(x) = \overline{G(-x)}$, then $G * \Gamma = F$, because Fourier transform takes convolutions into products.

Hence

$$\phi(0) = \int F(x)dx \leq \|G\|_1 \|\Gamma\|_1 = |\int_{-1/2}^{1/2} G(x)\, dx|^2 \leq \int_{-1/2}^{1/2} |G(x)|^2\, dx = F(0)$$

proving our claim. Accidentally, we also prove that if $F(0) = \phi(0)$ then F is a multiple of the triangle function considered above. $\square$

CHAPTER 5

A bound for the second moment

In this section we show that we have the following.

THEOREM 5.1. *Under the RH for $L(s, f)$ and the Linnik-Selberg conjecture we have the bound*

$$\sum{}' \frac{r_f^2}{4\pi\langle f, f\rangle} \leq \frac{31}{12} + o(1)$$

as $N \to \infty$.

PROOF. Square the formula

$$r_f \log x \leq \log N - 2\sum_{n\leq x} \frac{c_n(f)}{n}\Lambda(n)(1 - \frac{\log n}{\log x}) + O(1)$$

to get

$$\begin{aligned} r_f^2 \log^2 x \leq (\log N + O(1))^2 \\ + 4\sum_{n,\, m\leq x} \frac{c_n(f)c_m(f)}{nm}\Lambda(n)(1 - \frac{\log n}{\log x})\Lambda(m)(1 - \frac{\log m}{\log x}) \\ - 4(\log N + O(1))\sum_{n\leq x} \frac{c_n(f)}{n}\Lambda(n)(1 - \frac{\log n}{\log x}). \end{aligned}$$

Therefore

$$\sum{}' \frac{r_f^2 \log^2 x}{4\pi\langle f, f\rangle} \le \log^2 N + O(\log N)$$

$$+4 \sum_{n,\, m\le x} \Lambda(n)\Lambda(m)(1-\frac{\log n}{\log x})(1-\frac{\log m}{\log x}) \sum{}' \frac{c_n(f)c_m(f)}{nm4\pi\langle f, f\rangle} \tag{5.1}$$

$$-4(\log N + O(1)) \sum_{n\le x} \sum{}' \frac{c_n(f)}{n4\pi\langle f, f\rangle}\Lambda(n)(1-\frac{\log n}{\log x}). \tag{5.2}$$

Notice that (5.2) has been studied in [**Mur**$_1$] already so we need consider only (5.1).

Lemma 5.1. *Under Linnik-Selberg, we have*

$$4 \sum_{n,\, m\le x} \Lambda(n)\Lambda(m)(1-\frac{\log n}{\log x})(1-\frac{\log m}{\log x}) \sum{}' \frac{c_n(f)c_m(f)}{nm4\pi\langle f, f\rangle} = \frac{7\log^2 x}{12} + o(\log^2 x)$$

provided $x = o(N)$.

Proof of lemma 5.1. We split up the sum into six parts $\sum_{11}, \cdots, \sum_{33}$, where the double index corresponds to the power of the prime dividing n (resp. m).

By the bound (2.3) we have

$$\sum{}_{33} \ll \sum_{\substack{a,b\ge 3\\ p^a,\ q^b\le x}} \frac{\log p\ \log q}{p^{a/2}q^{b/2}}(1+N^{-2}) \ll 1.$$

To study $\sum_{11}$ we have to prove

Lemma 5.2. *Let F be a continuous, compactly supported function, or a $C^1(\mathbb{R})$ function with $F(u) = o(1/u)$, $uF'(u)$ and $uF(u)$ in $L^1(\mathbb{R})$, then*

$$\lim_{x\to\infty} \frac{1}{\log^2 x} \sum_p \frac{\log^2 p}{p} F(\frac{\log p}{\log x}) = \int_0^\infty uF(u)\,du$$

Proof of lemma 5.2. This is proven in much a similar way to lemma 4.4. Indeed by partial summation

$$\begin{aligned}
&\sum_p \frac{\log^2 p}{p} F(\frac{\log p}{\log x}) \\
&= -\int_1^\infty (\sum_{p\le t} \frac{\log^2 p}{p}) F'(\frac{\log t}{\log x}) \frac{dt}{t\log x} \\
&= -\int_1^\infty (\frac{\log^2 t}{2} + O(\log t)) F'(\frac{\log t}{\log x}) \frac{dt}{t\log x} \\
&= -\int_0^\infty \frac{u^2}{2} \log^2 x \; F'(u)\, du + O(\|uF'(u)\|_1 \log x) \\
&= \log^2 x \int_0^\infty uF(u)\, du + O(\|uF'(u)\|_1 \log x).
\end{aligned}$$

□

By taking $F(u) = \max(1-|u|, 0)$ we get

$$\int_0^\infty uF(u)\, du = \int_0^1 u(1-u)^2\, du = \frac{1}{12}$$

Hence

$$\begin{aligned}
\sum\nolimits_{11} &= \sum_{p,\, q\le x} \frac{\log p \log q}{pq} (1-\frac{\log p}{\log x})(1-\frac{\log q}{\log x}) \sum{}' \frac{\hat{f}(p)\hat{f}(q)}{4\pi\langle f, f\rangle} \\
&\ll \sum_{p\le x} \frac{\log^2 p}{p} (1-\frac{\log p}{\log x})^2 + N^{-2} \sum_{p,q\le x,\, p\ne q} \log p \; \log q \\
&+ N^{-2} \sum_{p\le x} \sqrt{p} \log^2 p \\
&\ll \frac{\log^2 x}{12} + (\frac{x}{N})^2 + \frac{x^{3/2}\log x}{N^2}.
\end{aligned}$$

To deal with $\sum_{22}$ we need the following lemma.

LEMMA 5.3. *We have*

$$\sum{}' \frac{c_{p^2}(f)c_{q^2}(f)}{4\pi\langle f,f\rangle} = (\delta_{pq}+1)pq + O(N^{-2}p^2q^2(p,q))$$

PROOF. Write $c_{p^2}(f) = \hat{f}(p^2) - p$, similarly for q, expand the $\sum{}'$ and use the quasi-orthogonality. □

We have by lemmas 4.4 and 5.3

$$\begin{aligned}\sum{}_{22} &= \sum_{p^2,\,q^2\le x} \frac{\log p\log q}{p^2q^2}\Big(1-\frac{2\log p}{\log x}\Big)\Big(1-\frac{2\log q}{\log x}\Big)\sum{}' \frac{c_{p^2}(f)c_{q^2}(f)}{4\pi\langle f,f\rangle}\\ &= \sum_{p^2,\,q^2\le x} \frac{\log p\log q}{pq}\Big(1-\frac{2\log p}{\log x}\Big)\Big(1-\frac{2\log q}{\log x}\Big)\\ &+ \sum_{p^2\le x}\frac{\log^2 p}{p^2}\Big(1-\frac{2\log p}{\log x}\Big)^2\\ &+ O\Big(N^{-2}\sum_{p^2,\,q^2\le x}\log p\log q + N^{-2}\sum_{p^2\le x} p\log^2 p\Big)\\ &= \frac{\log^2 x}{16} + O(N^{-2}x\log x).\end{aligned}$$

We want to prove now that the cross-terms do not contribute significantly. Indeed

$$\begin{aligned}\sum{}_{21} &= \sum_{p^2,\,q\le x} \frac{\log p\log q}{p^2q}\Big(1-\frac{2\log p}{\log x}\Big)\Big(1-\frac{\log q}{\log x}\Big)O(N^{-2}p^2q(p,q)^{1/2})\\ &\ll N^{-2}\sum_{p^2,\,q\le x}\log p\log q + N^{-2}\sum_{p^2\le x}\sqrt{p}\log^2 p)\\ &\ll N^{-2}x^{3/2}.\end{aligned}$$

Also, by (2.3)

$$\sum\nolimits_{32} = \sum_{\substack{p^a,\, a\geq 3\\ q^2\leq x}} \frac{\log p \log q}{p^a q^2}\Big(1-\frac{a\log p}{\log x}\Big)\Big(1-\frac{2\log q}{\log x}\Big)\sum\nolimits' \frac{c_{p^a}(f)c_{q^2}(f)}{4\pi\langle f, f\rangle}$$
$$\ll \log x.$$

$\sum_{31}$ is a little more difficult to estimate. We begin with a lemma.

LEMMA 5.4. *Define the multiplicative function $\alpha(n)$ by the identity*

$$\frac{1}{L(s,f)} = \sum_{n\geq 1}\frac{\alpha(n)}{n^s}$$

$$\textit{Then}\begin{cases} \alpha(p) = -\hat{f}(p), & \\ \quad\alpha(p^2) = p & \textit{if } p\nmid N,\\ \quad\alpha(p^2) = 0 & \textit{if } p\mid N,\\ \quad\alpha(p^a) = 0 & \textit{if } a\geq 3.\end{cases}$$

PROOF. We use the Euler product

$$L(s,f) = \prod_{p\mid N}\frac{1}{1-\hat{f}(p)p^{-s}}\prod_{p\nmid N}\frac{1}{1-\hat{f}(p)p^{-s}+p^{1-2s}}$$

which implies that

$$\sum_{n\geq 1}\frac{\alpha(n)}{n^s} = \prod_{p\mid N}(1-\hat{f}(p)p^{-s})\prod_{p\nmid N}(1-\hat{f}(p)p^{-s}+p^{1-2s})$$

and hence the result. □

Going back to $\sum_{31}$ we have

$$\sum_{31} = \sum_{\substack{a\geq 3\\ p^a\leq x\\ q\leq x}} \frac{\log p \log q}{p^a q}(1-\frac{a\log p}{\log x})(1-\frac{\log q}{\log x})\sum{}' \frac{c_{p^a}(f)\hat{f}(q)}{4\pi\langle f,f\rangle}$$

$$= \sum_{\substack{a\geq 3\\ p^a\leq x\\ p\neq q\leq x}} \frac{\log p \log q}{p^a q}(1-\frac{a\log p}{\log x})(1-\frac{\log q}{\log x})$$

$$\times \sum{}' \sum_{d|p^a} \frac{\alpha(\frac{p^a}{d})\hat{f}(d)\hat{f}(q)\log d}{4\pi\langle f,f\rangle \log p} + O(1)$$

$$\ll N^{-2}\log x \sum_{\substack{a\geq 3\\ p^a\leq x\\ p\neq q\leq x}} \frac{\log p\log q}{p^a}\sum_{d|p^a} d\left|\alpha(\frac{p^a}{d})\right|$$

$$\ll N^{-2}\log x \sum_{\substack{a\geq 3\\ p^a\leq x\\ p\neq q\leq x}} \log p \log q$$

$$\ll N^{-2}x^{4/3}\log^2 x.$$

This concludes the proof of lemma 5.1. □

To conclude with the proof of the theorem note that (5.2) is majorized by $\log N \log x$ + lesser terms, hence taking $x = N^{1-\epsilon}$ gives the desired result.

□

CHAPTER 6

Improving the bound on the rank of $J_0(N)$

6.1. Sum of traces

For simplicity we will suppose that N is prime. In this case one has the following lemma (cf [**Van**]).

LEMMA 6.1 (VanderKam). *Let (w_n) be any sequence of numbers. If N is prime and $B < N^{2-\delta}$ for some $\delta > 0$, then for all $c > 1$*

$$\sum_{n=B}^{cB} w_n t_n = \frac{N}{12} \sum_{n=B}^{cB} \frac{w_n}{\sqrt{n}} \delta(n) + w_{max} O(B^{7/4} N^{-1/2} + B^{39/32+\epsilon} N^{1/4})$$

where

$$\begin{cases} t_n & = \operatorname{Trace}(\mathbf{T}_n \mid S_2(N)) \\ w_{max} & = \max\limits_{B \le n \le cB} |w_n| \end{cases}$$

and $\delta(n)$ is 1 if n is a square and zero otherwise.

COROLLARY 6.1. *If $x < N^{2-\delta}$ we have*

$$\sum_{p \le x} t_p \ll x^{7/4} (\log x)\ N^{-1/2} + x^{39/32+\epsilon} (\log x)\ N^{1/4}.$$

PROOF. Apply the lemma to estimate the sum of primes between x and $x/2$, then between $x/2$ and $x/4$ etc., always bounding them by

$$x^{7/4} N^{-1/2} + x^{39/32+\epsilon} N^{1/4}$$

Since there are at most $\log x / \log 2$ sums we are done. □

Now from (4.1) with $F(t) = \max(1 - |t|, 0)$ we get

$$\sum{}' r_f \leq \dim S_2(N) \frac{\log N}{\log x} - \frac{2}{\log x} \sum_{n \leq x} \Big(\sum{}' c_n(f) \Big) \frac{\Lambda(n)}{n} \Big(1 - \frac{\log n}{\log x}\Big) + O\Big(\frac{N}{\log x}\Big).$$

Then again decompose the double sum into $\sum_1 + \sum_2 + \sum_3$ and use the same bounds on $\sum_2$ and $\sum_3$. VanderKam's lemma will allow us to find a less restrictive bound on $\sum_1$ so that we will be able to improve on Murty and Brumer's bounds.

Indeed notice that

$$\sum{}_1 = \sum_{p \leq x} t_p \frac{\log p}{p} \Big(1 - \frac{\log p}{\log x}\Big).$$

By partial summation

$$\begin{aligned} \sum_{p \leq x} t_p \frac{\log p}{p} &= \frac{\log x}{x} \sum_{p \leq x} t_p - \int_2^x \Big(\sum_{p \leq t} t_p \Big) \, d\Big(\frac{\log t}{t}\Big) \\ &\ll \frac{\log x}{x} \sum_{p \leq x} t_p \\ &\ll x^{3/4} (\log^2 x) \; N^{-1/2} + x^{7/32+\epsilon} (\log^2 x) \; N^{1/4} \end{aligned}$$

provided $x \leq N^{2-\delta}$. Similarly we get

$$\sum_{p \leq x} t_p \frac{\log^2 p}{p} \ll x^{3/4} (\log^3 x) \; N^{-1/2} + x^{7/32+\epsilon} (\log^3 x) \; N^{1/4}$$

for $x \leq N^{2-\delta}$. Hence by taking $x = N^{2-\delta}$ for arbitrarily small $\delta > 0$ we get

Theorem 6.1. *Under the RH for $L(s, f)$ we get*

$$\limsup_{\substack{N \text{ prime} \\ N \to \infty}} \frac{\text{analytic rank } J_0(N)}{\dim S_2(N)} \leq 1.$$

6.2. Non-vanishing of $L(s, f)$ at the central point

It is interesting to apply the previous result to prove effective lower bounds on the proportion of $L(1, f)$ which are not zero.

COROLLARY 6.2.

(1) *Asymptotically, for at least* 25% *of forms* $f \in S_2(N)$ *we have that* $L(1, f) \neq 0$ *(N prime).*
(2) *Asymptotically, for at least* 25% *of forms* $f \in S_2(N)$ *we have that* $L(1, f) = 0$ *and* $L'(1, f) \neq 0$ *(N prime).*

PROOF. We use a parity argument which follows closely the reasoning of section 1.3.3. Recall that if $f \in S_2^+(N)$ then $\operatorname{ord}_{s=1} L(s, f)$ is odd and if $f \in S_2^-(N)$ then $\operatorname{ord}_{s=1} L(s, f)$ is even. Thus if a proportion c of $L(1, f)$ is zero, then $c \geq 1/2$ and the analytic rank is at least

$$\frac{1}{2}\dim S_2(N) + 2(c - \frac{1}{2})\dim S_2(N) + o(N).$$

Theorem 6.1 and this lower bound yield $c \leq 3/4$, whence the first part of the corollary. The second is dealt with similarly. □

Although finding good upper bounds on the analytic rank of $J_0(N)$ gives positive information on a proportion of $L(1, f)$ which are not zero, this is by no means the most effective method. Indeed Iwaniec and Sarnak, in [**I-S**$_2$] , have put forth a method which gives the same results unconditionally. Their method consists in finding asymptotic estimates of

$$\sum{}' m_f L(1, f) \qquad \text{and} \qquad \sum{}' m_f^2 L^2(1, f)$$

The choice of the mollifiers m_f is important here to get a positive proportion. Actually m_f depends linearly on a number of coefficients. Hence, by fixing the asymptotics of the first moment, the problem reduces to minimizing a quadratic form, subject to a linear condition. This method has its genesis in the work of Selberg [**Sel**$_2$] where he proves that a positive proportion of the zeros of $\zeta(s)$ lie on the critical line $\Re s = 1/2$. They actually prove the following.

THEOREM 6.2 (Iwaniec-Sarnak). *Let N be squarefree and such that $\phi(N) \sim N$. Asymptotically, at least 50% of the forms in $S_2^-(N)$ satisfy*

$$L(1, f) \geq \frac{1}{\log^2 N}$$

On average (over N), this is true for more than 50%.

By similar methods, Kowalski and Michel [**K-M**$_2$] obtained a proportion of 33%. Also, VanderKam [**Van**] finds a positive proportion using the crucial lemma we stated at the beginning of the chapter.

Regarding the odd order zeros, Kowalski and Michel proved in [**K-M**$_3$]

THEOREM 6.3 (Kowalski-Michel). *Asymptotically, at least 70% of the forms in $S_2^+(N)$ satisfy $L'(1, f) \neq 0$.*

REMARK. Iwaniec and Sarnak relate these non-vanishing results to the non-existence of Siegel zeros for $L(s, \chi)$ (cf. [**I-S**$_2$]).

CHAPTER 7

Higher moments of analytic ranks

In this chapter we will analyse the sums $\sum' r_f^k$ for modular L-series. The same kind of results can be deduced for the sums $\sum' r_\chi^k$ for Dirichlet L-series. In particular we prove

THEOREM 7.1. *Assume the RH for $L(s, f)$.*

Let N be an integer and $f \in S_2(N)$. If $r_f = \operatorname{ord}_{s=1} L(s, f)$ then

$$\sum{}' r_f^k \leq c_k \dim S_2(N) + o(N)$$

as $N \to \infty$, where c_k depends only on k.

It may be of interest to investigate about the value of c_k, although the present approach is unlikely to give a better bound than k^k. For example, if we could prove that $c_k = o(2^k)$, then Brumer's conjecture would follow.

7.1. Upper bounds for traces

We prove the main ingredient for the theorem.

LEMMA 7.1 (Multitrace lemma (easy part)). *Let $T > 1$. If $d_1 \cdots d_j$ is not a perfect square, then*

$$\sum{}' \hat{f}(d_1) \ldots \hat{f}(d_j) \ll N^{-1/2}(d_1 \ldots d_j)^{1+\epsilon} T + (d_1 \ldots d_j)^{1/2+\epsilon} N^{1+\theta} T^{-1/2}$$

where $\theta = 3/4$ unconditionally or $\theta > 0$ on the Lindelöf hypothesis for $L(s, \vee^2 f)$.

PROOF. The proof goes by induction on j. If $j = 1$ this is known by lemma 4.2. From the formula (1.6)

$$\hat{f}(n)\hat{f}(m) = \sum_{\substack{d|(n,m)\\(d,N)=1}} d\,\hat{f}(\frac{nm}{d^2})$$

we have

$$\begin{aligned}\sum{}' \hat{f}(d_1)\dots\hat{f}(d_j) &= \sum_{\substack{d|(d_1,d_2)\\(d,N)=1}} d\sum{}' \hat{f}(\frac{d_1d_2}{d^2})\hat{f}(d_3)\dots\hat{f}(d_j)\\ &\ll N^{-1/2}T(d_1\dots d_j)^{1+\epsilon/2}\sum_d \frac{1}{d^{1+\epsilon}}\\ &\quad + (d_1\dots d_j)^{1/2+\epsilon/2}N^{1+\theta}T^{-1/2}(d_1d_2)^{\epsilon/2}\end{aligned}$$

because $d(n) \ll n^{\epsilon}$. □

7.2. Studying $\sum' r_f^k$

Raise to the k-th power the inequality

$$r_f \log x \le \log N - 2\sum_{n\le x}\frac{c_n(f)}{n}\Lambda(n)(1-\frac{\log n}{\log x}) + O(1)$$

to get

$$r_f^k \log^k x \le \sum_{j=0}^{k}\binom{k}{j}(-2)^j\Big(\sum_{n\le x}\frac{c_n(f)\Lambda(n)}{n}(1-\frac{\log n}{\log x})\Big)^j(\log N + O(1))^{k-j}$$

We then are reduced to proving

$$\begin{aligned}&\sum_{n_1,\dots,n_j\le x}\Big(\sum{}' c_{n_1}(f)\dots c_{n_j}(f)\Big)\frac{\Lambda(n_1)}{n_1}\dots\frac{\Lambda(n_j)}{n_j}(1-\frac{\log n_1}{\log x})\dots(1-\frac{\log n_j}{\log x})\\ &\ll \dim(S_2(N))\log^j x.\end{aligned} \tag{7.1}$$

Note that each n_i is really a prime power. We split the outer sum according to whether one of the n_i equals a prime p not dividing the other indices. The remaining sum is bounded trivially. Indeed we have for the latter that

(1) either one of the n_i is a prime. In this case it will have to divide another index, which will be a power of n_i,

(2) or every n_i is a prime to a power strictly greater than 1.

In either case we use the trivial bound (2.3) and bound each $(1 - \log n_j / \log x)$ by 1. We get the desired bound because

$$\sum_{p \leq x} \frac{\log p}{p} \ll \log x \qquad \text{and} \qquad \sum_{p \leq x} \frac{\log^2 p}{p} \ll \log^2 x.$$

We deal with the former sum using the multitrace lemma. From the identity

$$-\frac{L'}{L}(s, f) = \sum_{n=1}^{\infty} \frac{c_n(f)}{n^s} \Lambda(n)$$

we get, notations as in lemma 5.4:

$$\Lambda(n_i) c_{n_i}(f) = \sum_{d_i | n_i} \alpha(\frac{n_i}{d_i}) \hat{f}(d_i) \log d_i$$

Note that we can use the multitrace lemma because by assumption $d_1 \ldots d_j$ is not a perfect square. Again we bound $(1 - \log n_j / \log x)$ by 1. Suppose for simplicity that $n_1 = p$ and $p \nmid n_i, i \neq 1$. Then formula (7.1) is bounded by

$$\sum_{\substack{p \leq x \\ p \nmid n_i,\, i>1 \\ n_2, \ldots, n_j \leq x}} \sum_{d_2 | n_2, \ldots, d_j | n_j} \alpha(\frac{n_2}{d_2}) \ldots \alpha(\frac{n_j}{d_j}) \frac{\log p \, \log d_2 \ldots \log d_j}{p \, n_2 \ldots n_j} \times$$
$$\times \sum{}' \hat{f}(p) \hat{f}(d_2) \ldots \hat{f}(d_j)$$

and from lemma 5.4

$$\ll \sum_{\substack{p\le x\\ p\nmid n_i,\, i>1\\ n_2,\dots,n_j\le x}} 4^j \frac{\log p}{p}\frac{\log n_2}{m_2}\dots\frac{\log n_j}{m_j}\times$$

$$\times \left(N^{-1/2}(p\,n_2\dots n_j)^{1+\epsilon}T + N^{1+\theta}T^{-1/2}(p\,n_2\dots n_j)^{1/2+\epsilon}\right)$$

$$\text{where, if } n_i = q^a, \begin{cases} m_i = n_i = q & \text{if } a=1,\\ m_i = q^{3/2} & \text{if } a=2,\\ m_i = q^{a-1} & \text{if } a\ge 3.\end{cases}$$

Hence if we choose $x = N^{\frac{3}{4k}-\varepsilon}$, $T = N^{\frac{3j}{4k}-\frac{\varepsilon}{2}}$, we get that (7.1) is $o(N)$ and that concludes the proof.

Remark. The proof also indicates that we can choose indeed $c_k = k^k$.

CHAPTER 8

The distribution of zeroes of L-functions

In this chapter we investigate about the location of the first zeroes of $L(s,f)$ and $L(s,\chi)$. We prove two results on the average order of magnitude of the first zero with positive ordinate.

8.1. Statement of results

As usual, assume the RH for $L(s,f)$. Let $f \in S_2(N)$.

THEOREM 8.1. *For any $c_1 > 0$ there exists an effective positive constant c_2 such that if*

$$A(N) = \#\Big\{|t| < \frac{c_1}{\log N} : L(1+it,f) = 0 \quad \text{for some } f \in S_2(N)\Big\}$$

then

$$A(N) < c_2 \dim S_2(N).$$

THEOREM 8.2. *There exist effective positive constants c_3 and c_4 such that if*

$$B(N) = \#\Big\{t \neq 0,\, |t| < \frac{c_3}{\log N} : L(1+it,f) = 0 \quad \text{for some } f \in S_2(N)\Big\}$$

then

$$B(N) > c_4 \dim S_2(N).$$

Likewise, similar statements exist for Dirichlet L-functions with q instead of N and $\varphi(q)$ instead of $\dim S_2(N)$. Since the proof is similar, mutatis mutandis, we shall focus on the case of modular forms.

8.2. Proofs

Proof of theorem 8.1. We start from the explicit formula, this time we take into account the contribution of all the zeros.

$$\sum{}' r_f \log x + \log x \sum{}' \sum_{t\neq 0:\ L(1+it,f)=0} \frac{4\sin^2(t\log\sqrt{x})}{(t\log x)^2} =$$

$$= \dim S_2(N)\log N - 2\sum_{n\leq x} \frac{\sum{}' c_n(f)}{n}\Lambda(n)(1-\frac{\log n}{\log x}) + O(N).$$

In particular

$$\sum{}' \sum_{|t|<\frac{c_1}{\log N}:\ L(1+it,f)=0} \frac{\sin^2(t\log\sqrt{x})}{(t\log\sqrt{x})^2} \ll_c \dim S_2(N)$$

for $x = N^c$ with $0 < c < 2$. Choose $c = 1/c_1$. Then

$$A(N)4\sin^2(1/2) \leq$$

$$\leq \sum{}' \sum_{|t|<\frac{c_1}{\log N}:\ L(1+it,f)=0} \frac{\sin^2(t\log N^{\frac{1}{2c_1}})}{(t\log N^{\frac{1}{2c_1}})^2}$$

$$\ll_c \dim S_2(N).$$

□

Proof of theorem 8.2.

Lemma 8.1. *There exists an even test function F, continuous, compactly supported in $\mathbb{R}$ such that the corresponding ϕ satisfies:*

(1) $\phi(0) = 0$,

(2) $\exists \xi_0 > 0 \colon \phi(\xi) \geq 0 \ \forall |\xi| < \xi_0$ *and* $\phi(\xi) \leq 0 \ \forall |\xi| > \xi_0$.

(3) $F(0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \phi(\xi)\, d\xi > 0$.

Proof. Let $\tilde{F}$ be an even regular function ($C^\infty(\mathbb{R})$) with compact support. Replacing it by $\tilde{F} * \tilde{F}$, we can also assume that its Fourier transform $\tilde{\phi}$ is nonnegative on real values of the variable. Then $-\tilde{F}''$ (resp. $\tilde{F}^{(4)}$) has Fourier transform $\xi^2 \tilde{\phi}(\xi)$ (resp. $\xi^4 \tilde{\phi}(\xi)$). Since clearly $-\tilde{F}''(0) > 0$, we get that $F = -\tilde{F}'' - \epsilon \tilde{F}^{(4)}$ has the required properties for a sufficiently small $\epsilon > 0$. $\square$

We then apply the explicit formula to F and get

$$\log x \sum{}' \sum_{t \neq 0 \colon L(1+it,f)=0} \phi(t \log x) =$$

$$= \dim S_2(N) F(0) \log N - 2 \sum_{n \leq x} \frac{\sum{}' c_n(f)}{n} \Lambda(n) F\Big(\frac{\log n}{\log x}\Big) + O(N)$$

Notice also that because of lemma 4.4 we have that

$$\sum_{n \leq x} \frac{\sum{}' c_n(f)}{n} \Lambda(n) F\Big(\frac{\log n}{\log x}\Big) = O(N)$$

as long as $x < N^{2-\epsilon}$. Hence

$$\sum{}' \sum_{t \neq 0 \colon L(1+it,f)=0} \phi(t \log x) = \dim S_2(N) F(0) \frac{\log N}{\log x} + o(N)$$

when $x = N^c$ for $0 < c < 2$. Define $c_3 = \xi_0$ as in the lemma and take $x = N$. Then

$$\begin{aligned} & B(N) \max_{|\xi| < \xi_0} \phi(\xi) \geq \\ & \geq \sum{}' \sum_{t \neq 0 \colon L(1+it,f)=0} \phi(t \log x) \gg \\ & \gg \dim S_2(N). \end{aligned}$$

$\square$

Remark. Let t be a positive real that is the ordinate of a zero of $L(s, f)$, where $f \in S_2(N)$, and $t \log N < c$. It is interesting to ask if the numbers $t \log N$ are uniformly distributed in $(0, c]$ with respect to some continuous density measure. If this is the case, then the inequality in theorem 6.1 is strict, as long as $c > \pi$.

Also, is the density measure always non-zero? Partial results by Mestre seem to indicate that if $c < 1/10$ then the density measure should be identically zero (cf. [**Mes**, p. 221]).

CHAPTER 9

The order of vanishing on the critical line

In this chapter we give one more application of the explicit formula method to get, under RH for $L(s,f)$, that on average 50% of all $L(1+ia,f)$ do not vanish, where $a \neq 0$. Because, as Murty showed, the $L(s,f)$ are primitive, it is expected that they have very few common zeros. Hence the following conjecture.

CONJECTURE 7 (Ram Murty). *Let* $R_f(a) = \operatorname{ord}_{s=1+ia} L(s,f)$. *Then for any* $a > 0$

$$\sum{}' R_f(a) = o(\dim S_2(N)).$$

We prove a step towards this conjecture by sharpening the upper bound which we derived in chapter 6 for $a = 0$.

THEOREM 9.1. *Assume the RH for* $L(s,f)$. *If* $a > 0$*, then*

$$\limsup_{\substack{N\to\infty \\ N \text{ prime}}} \frac{\sum{}' R_f(a)}{\dim S_2(N)} \leq \frac{1}{2}.$$

COROLLARY 9.1. *Asymptotically when* N *is prime, 50% of all* $L(1+ia,f)$ *do not vanish.*

PROOF OF THEOREM. The idea to single out the order of vanishing of $L(s,f)$ at $s = 1+ia$ is again to use functions which approximate $(\delta_a + \delta_{-a})/2$, the delta distributions at a and $-a$. The inverse Fourier transform of this distribution is $\cos(au)$, hence we will apply the explicit formula in Theorem 2.3 to functions like

$\cos(au)F(u/\log x)$. Again the best choice for F comes from the triangle function $F(u) = \max(1-|u|, 0)$. With this choice of test function we get from (4.1)

$$\sum{}' R_f(a) \leq \dim S_2(N)\frac{\log N}{\log x} - \frac{2}{\log x}\sum_{n\leq x}(\sum{}' c_n(f))\frac{\Lambda(n)}{n}\cos(a\log n)(1-\frac{\log n}{\log x}) + O(\frac{N}{\log x}).$$

We proceed now similarly to chapter 4. Again we decompose the double sum into $\sum_1 + \sum_2 + \sum_3$. As before,

$$\sum_3 \ll \dim S_2(N).$$

We also estimate $\sum_1$ using the trace Lemma 4.2 in the improved form we gave in chapter 6, namely VanderKam's trace Lemma 6.1. This allows us to estimate

$$\sum_1 \ll \dim S_2(N),$$

as long as $x < N^{2-\delta}$. Similarly we can estimate the subsum in $\sum_2$ arising from the modular coefficients in the decomposition $c_{p^2}(f) = \hat{f}(p^2) - p$ using the trace Lemma 4.2, to get again an upper bound of $O(\dim S_2(N))$ when $x < N^{2-\delta}$.

The part with p is the main difference from the previous treatment. We state the result in a lemma.

Lemma 9.1. *Let $F\colon \mathbb{R} \to \mathbb{R}$ be a continuous even function with compact support, such that F' exists in $L^1(\mathbb{R})$ and $a \in \mathbb{R}$. Then*

$$\sum_p \frac{\log p}{p}\cos(2a\log p)F(\frac{2\log p}{\log x}) = \frac{\log x}{2}\int_0^\infty \cos(au\log x)F(u)\,du + O_a(\|F'\|_1),$$

uniformly for a in any bounded interval, the error term being regular with respect to a.

To prove this we first consider the case where F is the characteristic function of the interval $(-2, 2)$. In this case we have

LEMMA 9.2. *We have the following equality, for any $\nu > 1$:*

$$\sum_{p \leq x} \frac{\log p}{p} \cos(2a \log p) = \frac{\sin(2a \log x)}{2a} + H(x,a) + O((\log x)^{-\nu}),$$

where

$$H(x,a) = 2a \int_1^x \sin(2a \log t) O\left(\frac{1}{t \log^\nu t}\right) dt + \kappa,$$

and κ is an absolute constant (see below).

PROOF. This is done by partial summation after noting that another application of partial summation and the prime number theorem with explicit remainder yield

$$\sum_{p \leq x} \frac{\log p}{p} = \log x + \kappa + O\left(\frac{1}{\log^\nu x}\right)$$

for any $\nu > 0$, where κ is an absolute constant. Indeed we then have

$$\begin{aligned}\sum_{p \leq x} \frac{\log p}{p} \cos(2a \log p) &= (\log x + \kappa + O(\log^{-\nu} x)) \cos(2a \log x) \\ &+ \int_1^x (\log t + \kappa + O(\log^{-\nu} x)) \frac{2a \sin(2a \log t)}{t}\, dt \\ &= (\log x) \cos(2a \log x) + \int_1^x (\log t) \frac{2a \sin(2a \log t)}{t}\, dt \\ &+ H(x,a) + O\left(\frac{1}{\log^\nu x}\right).\end{aligned}$$

An integration by parts gives the desired result. □

Proof of Lemma 9.1. The proof is very similar to the proof of Lemma 4.4. By partial summation we have

$$\sum_p \frac{\log p}{p} \cos(2a \log p) F\left(\frac{2a \log p}{\log x}\right) =$$

$$= -\int_1^\infty \left(\sum_p \frac{\log p}{p} \cos(2a \log p)\right) F'\left(\frac{2 \log t}{\log x}\right) \frac{2\,dt}{t \log x}$$

$$= -\int_1^\infty \frac{\sin 2a \log t}{2a} F'\left(\frac{2 \log t}{\log x}\right) \frac{2\,dt}{t \log x} + O_a(\|F'\|_1).$$

Integrating by parts we get

$$\sum_p \frac{\log p}{p} \cos(2a \log p) F\left(\frac{2a \log p}{\log x}\right) =$$

$$= -\left[\frac{\sin(au \log x)}{2a} F(u)\right]_{u=0}^{u=\infty} + \frac{\log x}{2} \int_0^\infty \cos(au \log x) F(u)\,du + O_a(\|F'\|_1).$$

□

We end the proof of the theorem as in chapter 4, where the use of Lemma 9.1 allows us to remove a factor of $1/2$ (when $a \neq 0$) to get

$$\frac{\sum' R_f(a)}{\dim S_2(N)} \leq \frac{\log N}{\log x} + O\left(\frac{1}{\log x}\right),$$

provided $x < N^{2-\delta}$, thereby proving the theorem. □

Conclusion

This work is our attempt at discovering the connection between prime numbers and their analytic counterpart, that is the zeros of $L(s, f)$ or $L(s, \chi)$. The explicit formulas were the tools which enabled us to draw our conclusions on the growth of the order of vanishing and the distribution of the zeros on the critical line (n-correlation of the zeros) starting from the knowledge of the distribution of primes in $\mathbb{N}$. We think that these two notions constitute the two sides of one of the most difficult problems in analytic number theory. To explain the relationship between these notions in a unified view, let us define the generalised Tchebycheff ψ function for the modulus q or the level N. For simplicity we restrict ourselves to the case of prime q and N (otherwise use a similar reasoning as in chapters 3 and 4 to deal with the imprimitive part). In this case denoting $\boldsymbol{\omega} = \boldsymbol{\omega}_{\mathfrak{p}}$ to be either χ of modulus $\mathfrak{p}$ or $f \in S_2(\mathfrak{p})^{\text{new}}$, we can write

$$-\frac{L'}{L}(s, \boldsymbol{\omega}) = \sum_{n \geq 1} c_n(\boldsymbol{\omega}) \frac{\Lambda(n)}{n^s}$$

by logarithmic differentiation of the Euler product. We then define

$$\psi(x, \mathfrak{p}) = \sum_{n \leq x} \left(\sum_{\boldsymbol{\omega}}{}' c_n(\boldsymbol{\omega}) \right) \Lambda(n),$$

where $\sum'$ is for the sum over $\boldsymbol{\omega}$'s of same modulus $\mathfrak{p}$. The precise estimation of this generalised ψ is the key to future investigations to lower the upper bounds on

$\sum r_{\boldsymbol{\omega}}$. Let us note that ψ consists of a main term given by

$$\psi(x,\mathfrak{p}) = \frac{x}{\varphi(\mathfrak{p})} + E(x,\mathfrak{p})$$

in the case of Dirichlet series and by

$$\psi(x,\mathfrak{p}) = \frac{\mathfrak{p}\sqrt{x}}{12}\prod_{p|\mathfrak{p}}(1-p^{-2}) + E(x,\mathfrak{p})$$

in the case of modular forms. The main terms give rise to the conjectured estimations, hence the point is to have a good grip of $E(x,\mathfrak{p})$, which is precisely the content of VanderKam's lemma (Lemma 6.1).

It seems also possible, although the details are not clear to us yet, to obtain the same results as VanderKam's by using the Petersson formula on the quasi-orthogonality of Fourier coefficients of cusp forms given in Theorem 2.5. This could be done because

(1) Deshouillers and Iwaniec [**DI**] indicate that for a smoothed average of Kloosterman sums like in Theorem 2.5, the Linnik-Selberg conjecture is true, hence that (cf. [$\mathbf{Mur}_1$])

$$\sideset{}{'}\sum \frac{r_f}{4\pi\langle f,f\rangle} \leq 1 + o(1)$$

as $N \to \infty$,

(2) We would proceed as in [$\mathbf{Mur}_1$] by writing

$$L(2,\vee^2 f) = 8\pi^3\varphi(N)\frac{\langle f,f\rangle}{N^2},$$

to get rid of the harmonic weight and using the remark (already found in [$\mathbf{I\text{-}S}_2$, $\mathbf{K\text{-}M}_2$]) that on the average $L(2,\vee^2 f)$ can be represented by a Dirichlet series of very small length ($\ll N^\epsilon$), because it is the value of $L(s,\vee^2 f)$ at the edge of the domain of absolute convergence.

It would be interesting also to investigate whether the n-correlation conjectures would give us any new information on the distribution of primes. We would like to go in the direction of Heath-Brown as in [**HB**], or to see whether (assuming the RH for $L(s, \chi)$) we can improve on the Linnik result for the least prime in an arithmetic progression.

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